Zbl 1151.37056

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Soliton equations and their algebro-geometric solutions. Volume II. $(1+1)$-**dimensional discrete models.** (English)

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The monograph continues earlier work of Professors *F. Gesztesy* and *H. Holden* on the algebro-geometric solutions of hierarchies of soliton equations [see Vol. I, same series 79 (2003; Zbl 1061.37056)] associated with nonlinear partial differential equations in one space and one time dimension. Enhancing the team that successfully worked on the first volume with two new members, in the second volume the authors concentrate their attention on completely integrable differential-difference equations dealing with the Toda, Kac-van Moerbeke and Ablowitz-Ladik hierarchies which read as follows:

$$\text{Tl:}\quad \begin{pmatrix} a_t - a\left(b^+ - b\right) \\ b_t - 2\left(a^2 - \left(a^-\right)^2\right) \end{pmatrix} = 0,$$

$$\text{KM:}\quad \rho_t - \rho\left(\left(\rho^+\right)^2 - \left(\rho^-\right)^2\right) = 0,$$

$$\text{AL:}\quad \begin{pmatrix} -i\alpha_t - (1-\alpha\beta)\left(\alpha^+ + \alpha^-\right) + 2\alpha \\ -i\beta_t - (1-\alpha\beta)\left(\beta^+ + \beta^-\right) - 2\beta \end{pmatrix} = 0,$$

where the notation $\phi^{\pm}$ stands for the shift of a lattice function ϕ defined by $\phi^{\pm}(n) = \phi(n \pm 1)$, $n \in \mathbb{Z}$.

The main purpose of the monograph is to develop efficient methods for a simultaneous construction of all algebro-geometric solutions for some classes of completely integrable nonlinear differential-difference equations along with their theta function representations of hierarchies. Algebro-geometric solutions represent a natural extension of soliton solutions and similar to these; they also can be used to approximate more general solutions as, for instance, almost periodic ones. Starting with a given integrable differential-difference equation, one can build an infinite sequence of higher-order differential-difference equations, the so-called hierarchy of the original solution equation, by developing an explicit recursive formalism that reduces the construction of the entire hierarchy to elementary manipulations with polynomials and defines the associated Lax pairs of zero-curvature equations. Using this recursive polynomial formalism, one simultaneously constructs algebro-geometric solutions for the entire hierarchy of solutions.

The volume is very well structured, the exposition is transparent and accurate. There are three chapters dealing, respectively, with the Toda, Kac-van Moerbeke and Ablowitz-Ladik hierarchies. In the first chapter, algebro-geometric solutions to the Toda hierarchy are studied. The integrability of the Kac-van Moerbeke equation is Chapter 2 is explored through its intrinsic connections with the Toda lattice by means of a Miura type transformation which allows transfer of solutions between two equations. Finally, Chapter 3 addresses algebro-geometric solutions of a complexified discrete nonlinear Schrödinger hierarchy, the Ablowitz-Ladik hierarchy.

Similarly to the first volume, the authors strive towards making the presentation self-contained at the price of repeating in each chapter similar arguments regarding construction of algebro-geometric solutions to different hierarchies. On the positive side, one can read each chapter independently from the material presented in the remaining parts of the book. Significant efforts have been made to achieve a very neat presentation, sufficiently detailed, maximally explicit and precise. Although two volumes are independent of each other, notation in the second volume is consistent, whenever possible, with that introduced in the first one. To simplify the presentation, the authors have chosen not to employ advanced tools such as loop groups, Lie algebras, Grassmannians, formal pseudo-differential expressions, etc. Instead, they use fundamental methods from the theory of differential-difference equations, elements of algebraic geometry (in particular, the basic theory of compact Riemann surfaces), and some spectral analysis. Additional comments with bibliographic notes at the end of each chapter complement well the material and often include suggestions for further reading. Three nice appendices collect many useful facts on algebraic curves and their theta functions, hyperelliptic curves of the Toda type, asymptotic spectral parameter expansions and nonlinear recursion relations and can be also used as independent reference material. The list of references on the subject is extensive and reflects numerous contributions to the field made over the past four decades. The last but not least: the selection of epigraphs to the sections is delightful.
This skillfully written text describing the state-of-art developments in one of the important brunches of mathematical and theoretical physics is a very useful reference for graduate students and researchers interested in the theory and applications of completely integrable systems.

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Keywords : differential-difference equation; algebro-geometric solutions; integrability; Riemann theta function; algebraic curves; hierarchy; solitons
Classification :
∗37K40 Soliton theory, asymptotic behavior of solutions
14K25 Theta-functions
37-02 Research exposition (Dynamical systems and ergodic theory)
34Kxx Functional-differential equations, etc.
35Q51 Solitons
37K10 Completely integrable systems etc.
37K20 Relations with algebraic geometry, etc.