Errata and Addenda

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Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the kth line from the top (resp., the bottom) of a page.

Introduction

Page 10. In equation (0.34), it should read:

$$P_{2n+1}$$

Page 15. In line 3+, it should read:

$$P_{2n+1}$$

Chapter 1

Page 28. In line 8+, the second term should read:

$$\big(u_{t_n} + (L-z)P_{2n+1}\big)\big|_{\ker(L-z)} = 0,$$

Page 32. In equation (1.46) and in line 4−, it should read:

$$P_{2n+1}$$

Page 32. In equation (1.52), it should read:

$$F_n(z,x')$$

Page 32. In lines 3− and 2−, it should read: Equation (1.48) follows by combining (1.41) and (1.44).

Page 35. In line 9+, it should read:

$$\underline{\hat{\lambda}}^{\beta,\ell} = \{\hat{\lambda}_0^\beta, \hat{\lambda}_1^\beta, \dots, \hat{\lambda}_{\ell-1}^\beta, \hat{\lambda}_{\ell+1}^\beta, \dots, \hat{\lambda}_n^\beta\}, \quad \ell = 2, \dots, n-1,$$

Page 39. Equation (1.79) should have only one factor of i on the right-hand side, so the i to the right of $\sum_{j=1}^{n}$ should be stricken, that is, equation (1.79) should read

$$F_{n,x}(z) = 2i \sum_{j=1}^{n} y(\hat{\mu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} (z-\mu_k)(\mu_j-\mu_k)^{-1},$$

Page 40. Equation (1.80) should have $+F_{n,x}(z)$ on the right-hand side, that is, it should read

$$\partial_\beta K_{n+1}^{\beta}(z) = F_{n,x}(z) + 2\beta F_n(z)$$

Similarly, the first line of (1.81) should have $+F_{n,x}\big(\lambda_\ell^\beta\big)$ on the right-hand side.

Page 41. In line 2+, it should read:

...with (1.5), (1.11), and (1.16) taken into account.

Page 41. In line 9+, it should read:

...Relations (1.85) and (1.86)...

Page 41. In line 10+, it should read:

...for K_{n+1}^{β} with (1.16) and (1.56) taken into account.

Page 41. In Lemma 1.18 it suffices to assume $u \in C^\infty(\mathbb{R})$. The additional assumption $u \in L^\infty(\mathbb{R})$ is superfluous since for each fixed $x \in \mathbb{R}$, $|\mu_j(x)|$ is finite and hence $\hat{\underline{\mu}}_j(x)$ cannot coincide with the branch point P_∞ (a fact used in the last part of the proof on p. 42). Analogously, it suffices to assume that u satisfies Hypothesis 1.33 in the time-dependent setting.

Page 42. In line 12+, it should read:

$$\lim_{x\to x_0} \hat{\mu}_{j_p}(x) = (\mu_0, -(i/2)F_{n,x}(\mu_0,x_0)), \quad p=1,\dots,N;$$

Page 44. Line 5− should read:

One infers from (1.98) that

Page 44. Equation (1.100) should read:

$$\int_{Q_0}^{P} \omega_{P_\infty,0}^{(2)} \underset{\zeta\to 0}{=} -\zeta^{-1} + e_{0,0} + O(\zeta) \text{ as } P \to P_\infty$$

for some $e_{0,0} \in \mathbb{C}$.

Page 44. In lines 1−, 2−, and 3−, strike the sentences

since by (1.98),... sheets $\Pi_\pm$. Thus, ... contains no constant term.

Page 45. In equation (1.105), it should read:

$$\cdots \exp\bigg(-i(x-x_0)\bigg(\int_{Q_0}^{P} \omega_{P_\infty,0}^{(2)} - e_{0,0} \bigg)\bigg)$$

Page 52. In equation (1.128) and in the line following it, replace $\mathbb{Z}^n \setminus \{0\}$ by $(\mathbb{Z}^n \setminus \{0\})\tau$

Page 52. In the second line following equation (1.128), replace $i\Omega U_{0,j}^{(2)} = m_j$ by $i\Omega U_{0,j}^{(2)} = m_1\tau_{1,j} + \sum_{k=2}^{n}(m_k - m_{k-1})\tau_{k,j}$

Page 52. In the fifth line following equation (1.128), it should read:

…interval of length Ω, $x \in [x_0, x_0 + \Omega]$, for some $x_0 \in \mathbb{R}$.

Pages 57–63. There is a systematic error in Examples 1.30–1.32: All formulas for y_j should be replaced by iy_j, $j = 1, 2$. In Example 1.30, the quantities $\mathcal{F}_k(z, y)$ should therefore be of the form $\mathcal{F}_k(z, y) = y^2 - R_{2k+1}(z)$ for $k = 1, 2, 3$, and $k = n$, for monic polynomials R_{2k+1} of the form z^3, z^5, z^7, and z^{2n+1}. In Example 1.31, $\mathcal{F}_k(z, y)$ should be of the form

$$\mathcal{F}_n(z, y) = y^2 - z \prod_{j=1}^{n} (z + \kappa_j^2)^2 = 0$$

and in Example 1.32 it should read

$$\mathcal{F}_1(z, y) = y^2 - \left(z^3 - \frac{g_2}{4} z + \frac{g_3}{4} \right) = 0$$

and

$$\begin{aligned} \mathcal{F}_2(z, y) &= y^2 - \left(z^5 - \frac{21}{4} g_2 z^3 - \frac{27}{4} g_3 z^2 + \frac{27}{4} g_2^2 z + \frac{81}{4} g_2 g_3 \right) \\ &= y^2 - (z^2 - 3g_2) \left(z^3 - \frac{9}{4} g_2 z - \frac{27}{4} g_3 \right) = 0 \end{aligned}$$

Pages 61. In Example 1.31 assume $c_j, \kappa_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \dots, n$.

Pages 61. In Example 1.31 replace $\text{s-}\widehat{\text{KdV}}_n(u_n) = 0$ by

$$\text{s-KdV}_n(u_n) = 0$$

for an appropriate set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$ (cf. (1.15)).

Pages 66. Add the following to the end of the sentence following (1.164):

…except at collisions of certain μ_j (respectively, ν_ℓ), where one can only assert continuity of μ_j (respectively, ν_ℓ) with respect to (x, t_r).

Page 76. Equation (1.207) should read:

$$\int_{Q_0}^{P} \widetilde{\Omega}_{P_\infty, 2r}^{(2)} \underset{\zeta \to 0}{=} -\sum_{q=0}^{r} \tilde{c}_{r-q} \zeta^{-2q-1} + \tilde{e}_{r,0} + O(\zeta) \text{ as } P \to P_\infty$$

for some $\tilde{e}_{r,0} \in \mathbb{C}$.

Page 77. In equation (1.211), it should read:

$$\cdots \exp\left(-i(x - x_0) \left(\int_{Q_0}^{P} \omega_{P_\infty, 0}^{(2)} - e_{0,0} \right) - i(t_r - t_{0,r}) \left(\int_{Q_0}^{P} \widetilde{\Omega}_{P_\infty, 2r}^{(2)} - \tilde{e}_{r,0} \right) \right)$$

Page 79. In line 9−, replace $-\underline{\widetilde{U}}_{2r}^{(2)}$ by $i\underline{\widetilde{U}}_{2r}^{(2)}$

Page 81. In lines 9− and 10−, it should read:

$$\cdots \exp\left(-i(x - x_0) \left(\int_{Q_0}^{P} \omega_{P_\infty, 0}^{(2)} - e_{0,0} \right) \right.$$

$$- i(t_r - t_{0,r})\Big(\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,2r} - \tilde{e}_{r,0}\Big) - \int_{Q_0}^{P} \omega^{(3)}_{\hat{\lambda}_0^\beta(x_0,t_{0,r}),\hat{\lambda}_0^\beta(x,t_r)}\Big)$$

Page 81. In line 4− replace . . . purely imaginary . . . by . . . real . . .
Pages 86. In Example 1.52 assume $c_j, \kappa_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \dots, n$.
Pages 86. Line 1− should read:

$$\text{s-KdV}_n(u_n) = 0, \quad \text{KdV}_r(u_n) = 0$$

for appropriate sets of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$ (cf. (1.15)) and $\{\tilde{c}_s\}_{s=1,\dots,r} \subset \mathbb{C}$.
Page 101. Line 2+ should start with: genus n. . . .
Page 102. In the last line of equation (1.282) and in equation (1.283), the remainder term can be replaced by:

$$O(|z|^{-n-(3/2)})$$

Page 102. In line 2 of equation (1.282), it should read:

$$\Big(\prod_{m=0}^{2n} (1 - (E_m/z))\Big)^{-1/2}$$

Page 102. In the last line of equation (1.284), the remainder term can be replaced by:

$$O(|z|^{-n-(5/2)})$$

Page 102. Line 4− should read:

$$\frac{\delta \hat{f}_\ell}{\delta u} = \frac{2\ell - 1}{2} \hat{f}_{\ell-1}, \quad \ell = 1, \dots, n.$$

Page 103. The last line of equation (1.287) should read:

$$= u_{t_n} - \partial_x(\nabla \mathcal{H}_n)_u,$$

Page 103. In line 2 of Theorem 1.62, it should read:

$$\dots \ell = 1, \dots, n \text{ for } n \in \mathbb{N} \dots$$

Page 117. In line 11+ replace $\tau\mathbb{Z}^n$ by $\mathbb{Z}^n\tau$
Page 141. Equation (2.106) should read:

$$\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} -\zeta^{-1} + e_{0,0} + O(\zeta) \text{ as } P \to P_\infty$$

for some $e_{0,0} \in \mathbb{C}$.
Page 142. In equation (2.111), it should read:

$$\cdots \exp\Big(- i(x - x_0)\Big(\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} - e_{0,0}\Big)\Big)$$

Page 142. In equation (2.112), it should read:

$$\cdots \exp\Big(-\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0} + \tfrac{1}{2}\ln(E_{m_0}) - i(x-x_0)\Big(\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} - e_{0,0}\Big)\Big)$$

Pages 151. Add the following to the end of the sentence following (2.159):
...except at collisions of certain μ_j (respectively, ν_k), where one can only assert continuity of μ_j (respectively, ν_k) with respect to (x, t_r).
Page 159. Equation (2.197) should read:

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r} \underset{\zeta\to 0}{=} \begin{cases} -\sum_{q=0}^{r-1} \tilde{c}_{r-1-q}\zeta^{-2q-1} + \tilde{e}_{r,0} + O(\zeta), & r\in\mathbb{N}, \\ 0, & r=0, \end{cases} \quad \text{as } P\to P_\infty$$

for some $\tilde{e}_{r,0}\in\mathbb{C}$.
Page 159. Equation (2.201) should read:

$$\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\int_{Q_0}^{P}\omega^{(2)}_{P_0,0} \underset{\zeta\to 0}{=} -\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\zeta^{-1} + d_0 + O(\zeta) \text{ as } P\to P_0$$

for some $d_0\in\mathbb{C}$.
Page 159. In lines 8−, 9−, 10− and 11−, strike the sentence
Since by (2.200), ...no constant term.
Page 160. In equation (2.204), it should read:

$$\begin{aligned} \cdots \exp\Big(&-i(x-x_0)\Big(\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} - e_{0,0}\Big) \\ &+ (t_r - t_{0,r})\Big(\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\int_{Q_0}^{P}\omega^{(2)}_{P_0,0} - d_0 + \int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,r} - \tilde{e}_{r,0}\Big)\Big) \end{aligned}$$

Page 160. In equation (2.205), it should read:

$$\begin{aligned} \cdots \exp\Big(&-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0} + (1/2)\ln(E_{m_0}) - i(x-x_0)\Big(\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} - e_{0,0}\Big) \\ &+ (t_r - t_{0,r})\Big(\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\int_{Q_0}^{P}\omega^{(2)}_{P_0,0} - d_0 + \int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,r} - \tilde{e}_{r,0}\Big)\Big) \end{aligned}$$

Chapter 2

Page 168. The second paragraph should read as follows:
If, in addition, $\ell = 0$ and one is interested in spatially periodic solutions u with a real period $\Omega > 0$, the additional periodicity constraints

$$i\Omega \underline{U}_0^{(2)} \in (\mathbb{Z}^n \setminus \{0\})\tau$$

must be imposed. (By (B.45) this is equivalent to $2i\Omega\underline{c}(n) \in (\mathbb{Z}^n\setminus\{0\})\tau$.)

Chapter 3

Page 180. Line 1−, should read:

$$Q_{n+1} = \sum_{\ell=0}^{n+1} c_{n+1-\ell} \widehat{Q}_{\ell}, \qquad \widehat{Q}_0 = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Page 181. In line 15+, replace KdV by AKNS.
Page 189. In line 4+, it should read $\check{h}_{n-\ell} = A h_{n-\ell}$.
Page 199. Delete r at the end of equation (3.108).
Page 204. In line 4−, it should read:
with non-self-adjoint Dirac-type operators ...
Page 212. In equation (3.146), it should read:

$$p(x, \cdot\,), q(x, \cdot\,) \in C^1(\mathbb{R})$$

Page 214. In line 8−, it should read:
... properties of F_n, G_{n+1}, H_n, ...
Pages 214. Add the following to the end of the sentence following (3.169):
... except at collisions of certain μ_j (respectively, ν_k), where one can only assert continuity of μ_j (respectively, ν_k) with respect to (x, t_r).
Page 222. In line 9−, replace $\widehat{F}_r$ by $\widetilde{F}_r$ twice.

Chapter 5

Page 302. In line 3+, replace P_∞ by P_{∞_+}.
Page 310. In line 7−, it should read:
... properties of F_n, G_n, H_n, ...
Pages 310. Add the following to the end of the sentence following (5.132):
... except at collisions of certain μ_j (respectively, ν_k), where one can only assert continuity of μ_j (respectively, ν_k) with respect to (x, t_r).

Appendix A

Page 329. In line 9−, it should read: ... The intersection of ...
Page 330. Line 10− should read:

$$\mathcal{P}(z, y, 1) = 0$$

Page 331. In line 6+, it should read:
... the ramification points). If ...
Page 337. In line 4+, it should read: ... is a smooth simple, ...
Page 347. In lines 7− and 6−, it should read:
... where $\{P_{\infty_1}, \dots, P_{\infty_N}\}$ (typically, $N \in \{1, 2\}$ in the main text), denotes the set of ...
Page 347. In line 2−, P_0 should be replaced by Q_0.
Page 354. In line 7−, it should read:
The case $\mathcal{R} \neq 0$, and ...

Appendix B

Page 363. In (B.39) replace $\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}dz}{R_{2n+1}(z)^{1/2}}$ by $\sum_{\ell=k}^{n} \int_{E_{2\ell-1}}^{E_{2\ell}} \frac{x^{j-1}dx}{R_{2n+1}(x)^{1/2}}$ and refer to the homology basis described on top of p. 360, recalling the ordering $E_0 < E_1 < \cdots < E_{2n}$.

Page 363. Equation (B.40) should read

$$\omega^{(3)}_{P_1,P_\infty} = \frac{y+y_1}{z-z_1}\frac{dz}{2y} + \frac{\lambda_n}{y}\prod_{j=1}^{n-1}(z-\lambda_j)\,dz, \tag{B.40}$$

Appendix C

Page 375. In (C.39) replace $\int_{E_{2k-2}}^{E_{2k-1}} \frac{z^{j-1}dz}{R_{2n+2}(z)^{1/2}}$ by $\int_{E_{2k-1}}^{E_{2k}} \frac{x^{j-1}dx}{R_{2n+2}(x)^{1/2}}$

Page 375. In (C.40) replace $\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}dz}{R_{2n+2}(z)^{1/2}}$ by $-\sum_{\ell=1}^{k} \int_{E_{2\ell-2}}^{E_{2\ell-1}} \frac{x^{j-1}dx}{R_{2n+2}(x+i0)^{1/2}}$ and refer to the homology basis indicated in Fig. C.2 on p. 37, changing all E_m into real position with the ordering $E_0 < E_1 < \cdots < E_{2n+1}$.

Appendix D

Page 383. In line 2+ it should read:

. . . coefficients of η^k yields

Page 385. In line 8−, it should read:

$\ldots c_\ell$, $\ell = 0, \ldots, n+1$,

Appendix E

Page 397. In the first displayed equation in the proof of Theorem E.1 and in equation (E.5), replace the subscript "j" by "k" twice. Thus, the first equation should read:

$$\frac{1}{2\pi i}\oint_{C_R} d\zeta\,\frac{\zeta^{m-1}}{F_n(\zeta)(\zeta - z)} = \frac{z^{m-1}}{F_n(z)} + \sum_{k=1}^{n}\frac{\mu_k^{m-1}}{F_{n,z}(\mu_k)(\mu_k - z)},$$
$$z \neq \mu_1, \ldots, \mu_n, \quad m = 1, \ldots, n+1,$$

and equation (E.5) should read:

$$z^{m-1} - \sum_{k=1}^{n}\frac{\mu_k^{m-1}F_n(z)}{F_{n,z}(\mu_k)(z-\mu_k)} = F_n(z)\delta_{m,n+1}. \tag{E.5}$$

Page 399. In line 2−, replace (E.14) by (E.13)

Appendix F

Page 404. In equation (F.18), it should read:

$$\sum_{\ell=0}^{n} d_{n,\ell}(\underline{E})\Phi_\ell^{(j)}(\underline{\mu})$$

Appendix J

Page 463. The first and third line in (J.36) for $M_{\alpha,1,1}(z,x_0)$ and $M_{\alpha,2,2}(z,x_0)$ should be interchanged, moreover it is possible to considerably simplify the formula for $M_{\alpha,2,2}(z,x_0)$ (the old $M_{\alpha,1,1}(z,x_0)$) as follows:

$$\begin{aligned} M_{\alpha,1,1}(z,x_0) &= i\tfrac{K^{\beta}_{n+1}(z,x_0)}{2(1+\beta^2)R_{2n+1}(z)^{1/2}}, \\ M_{\alpha,2,2}(z,x_0) &= i\tfrac{F_n(z,x_0)-\beta F_{n,x}(z,x_0)+\beta^2 H_{n+1}(z,x_0)}{2(1+\beta^2)R_{2n+1}(z)^{1/2}}. \end{aligned} \tag{J.36}$$

(The formula for $M_{\alpha,1,2}(z,x_0) = M_{\alpha,2,1}(z,x_0)$ in (J.36) is correct as it stands.)

List of Symbols

Page 468. In line 3+, it should read

$$\partial\widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$$

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Page 492. Replace Novikov, S. P. 1999 by Novikov, D. P. 1999

Addenda

Page 68. Lemma 1.35 is equivalent to:

$$-V_{n+1,t_r} + \big[\widetilde{V}_{r+1}, V_{n+1}\big] = 0.$$

Page 104. The following considerations are relevant in connection with KdV conservation laws: Assuming that $u = u(x,t_n)$ satisfies $u(\,\cdot\,,t_n) \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $t_n \in \mathbb{R}$ (for simplicity), we recall that the one-dimensional Schrödinger equation

$$L(t_n)\psi(z,\,\cdot\,,t_n) = z\psi(z,\,\cdot\,,t_n), \quad z \in \mathbb{C}\setminus\mathbb{R},\ t_n \in \mathbb{R}$$

has unique (up to constant multiples) Weyl–Titchmarsh solutions $\psi_\pm(z,\,\cdot\,,t_n)$ satisfying for all $R \in \mathbb{R}$,

$$\psi_\pm(z,\,\cdot\,,t_n) \in L^2([R,\pm\infty)), \quad z \in \mathbb{C}\setminus\mathbb{R},\ t_n \in \mathbb{R}.$$

The corresponding Weyl–Titchmarsh functions $m_\pm(z,x,t_n)$ are then defined by

$$m_\pm(z,x,t_n) = \partial_x \ln(\psi_\pm(z,x,t_n)).$$

One obtains an asymptotic spectral parameter expansion of $m_\pm(\,\cdot\,,x,t_n)$ as $z \to i\infty$ of the type (cf. (J.29), (J.30))

$$m_\pm(z,x,t_n) \underset{z\to i\infty}{=} \sum_{j=-1}^{\infty} m_{\pm,j}(x,t_n) z^{-j/2}, \tag{1}$$

and the Riccati equation (J.26) for $m_\pm(z,x,t_n)$ then implies the following recursion relations for the expansion coefficients $m_{\pm,j}$,

$$\begin{aligned} &m_{\pm,-1} = \pm i, \quad m_{\pm,0} = 0, \quad m_{\pm,1} = \mp\frac{i}{2}u, \quad m_{\pm,2} = \frac{1}{4}u_x, \\ &m_{\pm,j+1} = \pm\frac{i}{2}\bigg(m_{\pm,j,x} + \sum_{\ell=1}^{j-1} m_{\pm,\ell}m_{\pm,j-\ell}\bigg), \quad j = 2,3,\dots \end{aligned} \tag{2}$$

Moreover, we recall

$$m_{-,j} = (-1)^j m_{+,j}, \quad j \in \{-1\}\cup\mathbb{N}_0. \tag{3}$$

Theorem 1 *Suppose that $u \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ satisfies the nth KdV equation (1.287) (for some set of integration constants c_ℓ, $\ell = 1,\dots,n$, $n \in \mathbb{N}$). Then, the infinite sequence of KdV conservation laws takes on the form*

$$\partial_{t_n} m_{\pm,2\ell+1} = \partial_x\bigg(\sum_{k=0}^{n} c_{n-k}\sum_{p=0}^{k}\hat{f}_{k-p}m_{\pm,2\ell+1+2p}\bigg), \quad \ell \in \mathbb{N}_0. \tag{4}$$

Here $\hat{f}_\ell$ are the homogeneous coefficients (1.6). Similarly to the recursion relation (2) *for the coefficients $m_{\pm,j}$, the coefficients $\hat{f}_\ell$ can be computed recursively from (1.4) (putting all integration constants equal to zero) or directly from the nonlinear recursion relation (D.8).*

Proof The key to the derivation of (4) is the innocent looking identity

$$\partial_{t_n}(\partial_x \ln(\psi_\pm(z,x,t_n,)) = \partial_x(\partial_{t_n} \ln(\psi_\pm(z,x,t_n))),$$

or equivalently,

$$m_{\pm,t_n} = ((\ln(\psi_\pm))_{t_n})_x.$$

Assuming that $\psi_\pm(z,x,t_n)$ are chosen so that

$$\psi_{\pm,t_n} = P_{2n+1}\psi_{\pm,t_n} = F_n\psi_{\pm,x} - (1/2)F_{n,x}\psi_\pm \tag{5}$$

(cf. (1.18)) one obtains

$$m_{\pm,t_n} = \partial_x(F_n m_\pm - (1/2)F_{n,x}).$$

(That $\psi_\pm$ can be chosen to satisfy (5) has been discussed in [2].) Since by (1.11), $F_n = \sum_{k=0}^n c_{n-k}\sum_{p=0}^k \hat{f}_{k-p}z^p$, it suffices to consider the homogeneous case $\widehat{F}_n = \sum_{k=0}^n \hat{f}_{n-k}z^k$. One then obtains

$$\begin{aligned}
m_{\pm,t_n} &= \sum_{j=1}^\infty m_{\pm,j,t_n} z^{-j/2} \\
&= \partial_x(F_n m_\pm - (1/2)F_{n,x}) \\
&= \partial_x\left(\widehat{F}_n \sum_{j=-1}^\infty m_{\pm,j} z^{-j/2} - (1/2)\widehat{F}_{n,x}\right) \\
&= \partial_x\left(\sum_{k=0}^n \hat{f}_{n-k}z^k \sum_{j=-1}^\infty m_{\pm,j} z^{-j/2} - (1/2)\widehat{F}_{n,x}\right).
\end{aligned}$$

A comparison of powers of $z^{-j/2}$ then yields

$$m_{\pm,j,t_n} = \partial_x\left(\sum_{k=0}^n \hat{f}_{n-k} m_{\pm,j+2k}\right), \quad j \in \mathbb{N}.$$

Since every even order coefficient $m_{\pm,2\ell}$ is known to be a total derivative (i.e., $m_{\pm,2\ell} = \partial_x(\dots)$), the conservation laws associated with $m_{\pm,2\ell}$, $\ell \in \mathbb{N}$, are all trivial. The odd order coefficients $m_{\pm,2\ell+1}$ lead to a nontrivial infinite sequence of conservation laws. (Of course, by (5), $m_{+,2\ell+1}$ and $m_{-,2\ell+1}$, $\ell \in \mathbb{N}_0$, yield the same infinite sequence of KdV conservation laws.) □

The basic KdV functionals $\widehat{\mathcal{I}}_\ell = \widehat{\mathcal{I}}_\ell(u, u_x, u_{xx}, \dots, \partial_x^k u)$, are then given in terms of $\hat{m}_{+,2\ell+1}$ and $\hat{f}_{\ell+1}$ by

$$\widehat{\mathcal{I}}_\ell = i\int_{\mathbb{R}} dx\, \hat{m}_{+,2\ell+1}(x) = \frac{1}{2\ell+1}\int_{\mathbb{R}} dx\, \hat{f}_{\ell+1}(x), \quad \ell \in \mathbb{N}_0 \tag{6}$$

(cf. (1.268) and (1.285)). Equation (4) yields a direct proof of

$$\frac{d\widehat{\mathcal{I}}_\ell}{dt_n} = 0, \quad \ell \in \mathbb{N}_0,\ n \in \mathbb{N}$$

(cf. Theorem 1.62).

Real-valuedness of u is not essential for these considerations and can be dropped. Moreover, the decay assumptions on u as $|x| \to \infty$ can be considerably relaxed and replaced by the finiteness of certain moments of u.

This representation of the KdV conservation laws is perhaps simpler for computational purposes than the traditional one relying on the Lenard recursion operator. For a new twist to conserved KdV polynomials we refer to [4] (see also [1]).

Next, we supplement this particular addendum on KdV conservation laws by briefly sketching the extension of the Hamiltonian formalism to Bohr almost periodic KdV solutions in the space variable as discussed by Johnson and Moser [3]:

We start by noting that if f denotes a Bohr (uniformly) almost periodic function on $\mathbb{R}$, its ergodic mean $\langle f \rangle$ is given by

$$\langle f \rangle = \lim_{R \uparrow \infty} \frac{1}{2R} \int_{-R}^{R} dx\, f(x).$$

Suppose that u has the frequency module $\mathcal{M}(u)$. Then given a density F as on p. 97, one has

$$\mathcal{F}(u) = \lim_{R \uparrow \infty} \int_{-R}^{R} dx\, F\big(u, u_x, u_{xx}, \dots, \partial_x^m u\big) = \langle F(u) \rangle,$$

and assuming that the frequency module $\mathcal{M}(v)$ of v satisfies $\mathcal{M}(v) \subseteq \mathcal{M}(u)$, one obtains

$$\begin{aligned}
(d\mathcal{F})_u(v) &= \frac{d}{d\epsilon} \mathcal{F}(u + \epsilon v)\big|_{\epsilon=0} \\
&= \lim_{R \uparrow \infty} \int_{-R}^{R} dx \left(\sum_{k=0}^{m} (-\partial_x)^k \partial_{u^{(k)}} F v \right)(x) \\
&= \big\langle (\nabla \mathcal{F})_u v \big\rangle = \left\langle \frac{\delta F}{\delta u} v \right\rangle.
\end{aligned}$$

In analogy to p. 98, the Poisson brackets of two functionals $\mathcal{F}_1, \mathcal{F}_2$ are then given by

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \left\langle \frac{\delta F_1}{\delta u} \left(\partial_x \frac{\delta F_2}{\delta u} \right) \right\rangle.$$

Again one verifies that both the Jacobi identity as well as the Leibniz rule hold in this case. Moreover, if $\mathcal{F}$ is a smooth functional and u develops according to a Hamiltonian flow with Hamiltonian $\mathcal{H}$, that is,

$$u_t = (\nabla_s \mathcal{H})_u = \partial_x (\nabla \mathcal{H})_u = \partial_x \frac{\delta H}{\delta u},$$

then

$$\frac{d\mathcal{F}}{dt} = \frac{d}{dt} \langle F(u) \rangle = \{\mathcal{F}, \mathcal{H}\}.$$

Next, one introduces the fundamental function w by

$$w(z) = -\frac{1}{2}\left\langle \frac{1}{g(z,\,\cdot\,)} \right\rangle$$

for $|z|$ sufficiently large. Since

$$w'(z) = \langle g(z,\,\cdot\,)\rangle, \quad z \in \mathbb{C} \setminus \operatorname{spec}(H)$$

(cf. (1.266)), w extends analytically to $z \in \mathbb{C} \setminus \operatorname{spec}(H)$. One infers from $g = 1/(m_- - m_+)$ that

$$w(z) = \pm\langle m_\pm(z,\,\cdot\,)\rangle, \quad z \in \mathbb{C} \setminus \operatorname{spec}(H).$$

Here $m_\pm$ denote the half-line Weyl–Titchmarsh functions associated with H. The asymptotic expansion of $m_\pm$ as $|z| \to \infty$ has been recorded in (1), and its connection with the homogeneous coefficients $\hat{f}_\ell$ and hence with the KdV functionals $\widehat{\mathcal{I}}_\ell$ has been noted in (6). In particular, introducing

$$\widehat{I}_\ell = i\langle \hat{m}_{+,2\ell+1}\rangle = \frac{1}{2\ell+1}\langle \hat{f}_{\ell+1}\rangle, \quad I_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\widehat{I}_k, \quad \ell \in \mathbb{N}_0,$$

the KdV equations again take on the form

$$\mathrm{KdV}_n(u) = u_{t_n} - 4\partial_x(\nabla I_{n+1})_u = 0, \quad n \in \mathbb{N}_0.$$

Finally, one can show that $w(z_1)$ and $w(z_2)$ are in involution for arbitrary $z_1, z_2 \in \mathbb{C} \setminus \operatorname{spec}(H)$, and hence obtains

$$\{w(z_1), w(z_2)\} = 0, \quad z_1, z_2 \in \mathbb{C} \setminus \operatorname{spec}(H), \tag{7}$$
$$\{w(z), I_p\} = 0, \quad \{I_p, I_r\} = 0, \quad z \in \mathbb{C} \setminus \operatorname{spec}(H),\ p, r \in \mathbb{N}_0. \tag{8}$$

Naturally, these considerations apply to the special periodic case in which $\langle f\rangle$ for a periodic function f on $\mathbb{R}$ is to be interpreted as the periodic mean value.

Page 122. An extension of formula (1.299) already appeared on p. 428 in [3].

Page 145. Line 4− can be more effectively replaced by:
By equation (2.94) one concludes that

Page 182. Line 7+: It would have been more natural to write equation (3.18) as:

$$G_{n+1}(z) = \sum_{\ell=0}^{n+1} g_{n+1-\ell} z^\ell = \sum_{\ell=0}^{n+1} c_{n+1-\ell}\widehat{G}_\ell(z),$$

Page 198. We note that $\Omega_0^{(2)}$ in equation (3.96) has the explicit form

$$\Omega_0^{(2)} = \omega^{(2)}_{P_{\infty_+},0} - \omega^{(2)}_{P_{\infty_-},0} = \frac{z^n}{y}\sum_{k=0}^{1} c_{1-k}(\underline{E}) z^k dz + \frac{\tilde{\lambda}_n}{y}\prod_{j=1}^{n-1}(z-\tilde{\lambda}_j)dz,$$

where $c_k(\underline{E})$, $k \in \mathbb{N}_0$, are defined in (D.5) and $\tilde{\lambda}_j$, $j = 1, \dots, n$, are uniquely determined by the normalization

$$\int_{a_j} \Omega_0^{(2)} = 0, \quad j = 1, \dots, n.$$

This comment also applies to $\Omega^{(2)}_{\infty,0}$ in (4.215).

Pages 200 and 224. One uses the equality

$$\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}) = \underline{z}(P_{\infty_+}, \underline{\hat{\mu}})$$

(an elementary consequence of (3.58)) to compute the constant C in equations (3.113) and (3.224).

Page 220. We note that $\omega^{(2)}_{P_{\infty_+},q} - \omega^{(2)}_{P_{\infty_-},q}$ in equation (3.207) has the explicit form

$$\omega^{(2)}_{P_{\infty_+},q} - \omega^{(2)}_{P_{\infty_-},q} = \frac{z^n}{y}\sum_{k=0}^{q+1} c_{q+1-k}(\underline{E}) z^k dz + \frac{\tilde{\lambda}_n}{y}\prod_{j=1}^{n-1}(z-\tilde{\lambda}_j)dz, \quad q \in \mathbb{N}_0,$$

where $c_k(\underline{E})$, $k \in \mathbb{N}_0$, are defined in (D.5) and $\tilde{\lambda}_j$, $j = 1, \dots, n$, are uniquely determined by the normalization

$$\int_{a_j} \big(\omega^{(2)}_{P_{\infty_+},q} - \omega^{(2)}_{P_{\infty_-},q}\big) = 0, \quad j = 1, \dots, n.$$

This comment is also relevant in connection with (C.37).

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