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Gesztesy, Fritz (1-MO); **Holden, Helge** (N-NUST)

★**Soliton equations and their algebro-geometric solutions. Vol. I.**

$(1+1)$-dimensional continuous models.

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FEATURED REVIEW.

The concept of complete integrability for a partial differential equation (PDE) is still somewhat elusive [cf. What is integrability?, Springer, Berlin, 1991; MR1098334 (91k:58005)], even though almost 40 years have gone by and a staggering number of papers on the subject have been written since the days when experiments [cf. N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15 (1965), no. 6, 240–243] in Princeton led a group of mathematicians [C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Phys. Rev. Lett. 19 (1967), no. 19, 1095–1097] to suspect that the Korteweg-de Vries equation

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x = 0$$

has an infinite number of conservation laws. What is even more staggering is the amount of deep mathematical theory that was revealed, even newly developed, in exploring the "hidden symmetries" of (classes of) examples of PDEs that share similar properties, and were dubbed "soliton equations". The book at hand is all the more satisfying for this reason. This is a truly original approach to the complete integrability property, and one that removes the vagueness from the definition, yields complete answers in closed form to the greatest possible extent, explores large classes of equations that share the defining properties and gives the mathematical explanations of such subtle phenomena. The price for achieving all that is to restrict the analysis to the class of solutions produced by the function theory of a Riemann surface (a.k.a. "spectral curve"), the "algebro-geometric solutions" of the title. But this is not really a loss, for another reason that makes this book so strong a contribution: because the authors, although well-versed in complex function and Riemann surface theory, and although responsible for several of the algebro-geometric constructions in closed form that give completeness to the theory, are analysts at heart. The heart of their extensive contributions is spectral theory; as such, it does not stop at contemplating and classifying

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the "miracle" of the algebraic solution, but rather it recasts its deeper reasons into the general theory of ODEs and PDEs, among which the "integrable" ones are still vaguely defined. As such, this treatise is squarely placed in the midst of the general problem; brings out the spectral properties that are responsible for finiteness; and poses the far-reaching open question of identifying exactly what subclasses of ODEs or PDEs have the "complete integrability" property, in a fresh way that could yield the key to many remaining mysteries. Surely, this approach opens the doors to a vast and rich field of reasonable, precisely-posed open questions, which recommends the book to any graduate student, researcher who wishes to enter this field, or even expert who wishes to explore some specific feature of the solutions.

I'd like to elucidate these general comments on the field and the place of this particular work, by using an example which is to some extent peripheral to soliton equations, but is so classical (and beautiful in my view) that it allows me to show rather than tell. If you would open your well-thumbed Whittaker and Watson [E. T. Whittaker and G. N. Watson, A course of modern analysis, Reprint of the fourth (1927) edition, Cambridge Univ. Press, Cambridge, 1996; MR1424469 (97k:01072)] to subsection 23.41 you will see that solutions of the Lame equation

$$L = d^2/dx^2 - n(n+1)\wp(x) + \text{const}, \quad n \in \mathbb{N},$$

are expressed (up to change of variable) as a series $\sum_{r=0}^{\infty} b_r(x - e_2)^{n/2-r}$, where ${\wp'}^2 = 4\prod_{i=1}^{3}(\wp - e_i)$. The series terminates (becoming a Lame function) for very special choices of the constant. This kind of finiteness defines the spectral curve (by giving the ramification points, roughly speaking—I beg forgiveness for not stating all the technical assumptions in this review, and refer to the Gesztesy and Weikard article cited below). In fact, the spectral curve of the operator $L_c = d^2/dx^2 - c\wp(x)$ is algebraic ("finite-gap") precisely for the values $c = n(n+1)$. Equivalently, the ring of ordinary differential operators (ODOs) commuting with L_c is of genus greater than 0 (not simply a ring isomorphic to $\mathbb{C}[z]$), and its isospectral deformations are classified by the Jacobi variety of the spectral curve. The solution $u(x,t)$ of the KdV equation written as a compatibility condition $\partial_t L = [B, L]$, for the deformed Lame operator $L = \partial^2/\partial x^2 - u(x,t)$ and a third-order operator $B(x)$, is a g-soliton algebro-geometric solution (specifically, an elliptic soliton) if g is the genus of the spectral ring of ODOs that commute with L. The first author is responsible for the ground-breaking set of ideas that connected such criteria for finiteness of elliptic KdV solutions to a spectral problem for ODOs of a complex

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variable, Picard's theory (cf. [F. Gesztesy and R. Weikard, Bull. Amer. Math. Soc. (N.S.) 35 (1998), no. 4, 271–317; MR1638298 (99i:58075)] for a survey).

What is, then, the authors' unique viewpoint on algebro-geometric solutions of soliton equations? Let me use the KdV example to convey it. It is precisely the following Ansatz: KdV is defined by recursive time evolutions:

$$u_{t_n} = [P_{2n+1}, L] = 2f_{n+1,x}$$

where $f_0 = 1,\ f_{l,x} = -\frac{1}{4}f_{l-1,xxx} + uf_{l-1,x} + \frac{1}{2}u_x f_{l-1}$ are differential polynomials in the smooth, complex-valued function u. We know of course that this is a hierarchy obtained by deforming just one (formal) pseudo-differential operator in an infinite Grassmannian [cf. T. Miwa, M. Jimbo and E. Date, Solitons, Translated from the 1993 Japanese original by Miles Reid, Cambridge Univ. Press, Cambridge, 2000; MR1736222 (2001a:37109)], but the authors explicitly state that the book does not report that construction. The analyst's way to see these polynomials is rather to expand (in the spectral parameter) the Green's function—the kernel of the resolvent $(L-\lambda)^{-1}$. The solutions are algebro-geometric if and only if a constant-coefficient combination of a finite number of time evolutions is zero, and the authors transparently interpret this Ansatz in terms of the Burchnall-Chaundy theory, which they call "stationary" KdV. The Ansatz produces an algebraic spectral curve, joint eigenfunctions have poles on a divisor that parametrizes isospectral deformations, and zeros on divisors whose coordinates satisfy the "Dubrovin equations"; trace formulae are available to recover the coefficients of the operators in terms of the theta function of the curve, by integrating the Dubrovin equations, both as dependent on x and on the time hierarchy t_n. Again, the analyst's way to see the auxiliary divisor (the zero of the joint eigenfunction, popularly known as Baker-Akhiezer function) is available in the special case when the potential $u(x)$ is periodic; poles and zeros are then natural normalizations of spectral data, namely the spectra for the Neumann and Dirichlet boundary conditions! [Cf. H. P. McKean, Jr., Comm. Pure Appl. Math. 38 (1985), no. 5, 669–678; MR0803254 (87c:14051).] This theory then covers both direct and inverse spectral problems for a Schrodinger operator $L = -d^2/dx^2 + u(x)$. Equivalently, the hierarchy is obtained as the zero-curvature condition

$$\partial_{t_n} U - \partial_x V_{n+1} + [U, V_{n+1}] = 0$$

where

$$U = \begin{bmatrix} 0 & 1 \\ -z + u(x, \underline{t}) & 0 \end{bmatrix}$$

and V_{n+1} is a suitable matrix depending on $x, \underline{t}, z$, and the finiteness comes down to a polynomial Ansatz in z on V_{n+1}, namely on the eigenvalues of a second ODO that commutes with L.

Part of what makes Riemann's theory of theta functions so beautiful is the interplay between algebra and geometry, the prototype being the cubic equation for the Weierstrass $\wp$ function. The authors' approach to soliton equations brings out that feature, by giving a thorough dictionary between the transcendental data (asymptotics of the solutions) and the algebraic. As an example of the power provided by the underlying spectral theory, the complete solution of Darboux/Backlund transformations for stationary KdV and AKNS hierarchy is given in terms of the Green's function, in Appendix G.

There is something of a precedent to this (spectral Ansatz) approach, namely a set of papers by S. I. Alber [J. London Math. Soc. (2) 19 (1979), no. 3, 467–480; MR0540063 (81b:35084)] on the polynomials, as well as work by R. Schimming [Acta Appl. Math. 39 (1995), no. 1-3, 489–505; MR1329579 (96b:35197)] on recursive equations for the polynomials by a kernel expansion of a heat operator. Neither of those authors, however, undertook a systematization of algebro-geometric solutions of soliton equations by recursion, but rather worked it out for basic examples such as KdV. The precedent is of course acknowledged, and this is a good place to say that the authors offer an incredibly thorough bibliography, both classical and contemporary (completeness of credits is part of the tradition of mathematical research in Austria [private communication from one of the authors]).

As the authors go through this neatly laid-out programme over and over, for the different PDEs of the five chapters of this volume, they cover deep geometric and analytic properties that are found nowhere else, including special classes of solutions (rational, solitons, elliptic) and specific statements on the reality conditions for the solutions.

The plan of the whole work is also well thought out. This is the first of three planned volumes; the theory is expounded through the main examples of soliton systems, each fundamentally different from the other, and ordered by increasing complexity, in this first volume as well as the prospective ones; the features that they share are laid out in parallel, the ones that are distinct are brought out. Volume I is concerned with continuum problems: KdV in Chapter I, sine-Gordon and modified KdV in Chapter II (the case of two points at infinity for the spectral curve, with a natural 2×2 matrix hierarchy of operators, which so subtly differs from the one-point case); the AKNS hierarchy in Chapter III (also 2×2 and including, by gauge transformation, the "classical" Boussinesq hierarchy, which is referred to as Kaup or Burgers in the literature). Chapter IV works out the algebro-geometric solutions for the classical massive Thirring system, of special importance in physics; the equations are not part of a hierarchy constructed recursively in the same way as the previous ones; the spectral hyperelliptic curve has two points over 0 and two over ∞; the theta function solution and reality conditions are given a careful analysis. Chapter V is devoted to the Camassa-Holm equation, a younger relative of KdV that carries a richer structure (the recursive definition of the polynomials entering the Lax-pair equations is non-local; its "peakon" solutions are solitons with unusual discontinuities) and is still a topic of active research; the authors have worked on the stationary theory. Volume II will be devoted to $(1+1)$ (hyperelliptic) lattice models, Volume III to the "non-hyperelliptic" case of the soliton hierarchies, culminating in the Davey-Stewartson case. Several papers by the authors and their collaborators have already indicated the computational and geometric difficulties to be ironed out in the non-hyperelliptic spectral-curve case.

At the end of each chapter, an overview commenting on the chapter's contents section by section provides historical insight into the development of the theory, including matters of transliteration from one school to another, as well as remarks on side aspects not covered in this book, further developments and open questions. The authors again do this in an astonishingly complete way, generously distributing credit for various contributions through the complex and vast unfolding of the theory. At the end of the book, ten clear, comprehensive appendices provide the needed technical tools from scratch (without proofs). These are the topics covered: theta functions, hyperelliptic curves of the KdV type, hyperelliptic curves of the AKNS type, asymptotic expansions, Lagrange interpolation, symmetric functions, Darboux transformations, elliptic functions, Herglotz functions (produced by a Schwarz reflection principle and important in spectral theory), and expansion of Weyl-Titchmarsh functions (providing a kind of spectral monodromy for eigenfunctions of the Schrodinger operator); it is rare to find such a collection of topics, especially worked out to be applied to all the hierarchies that are treated in the book.

As for the style, the 18-page introduction reveals in a progressive and lucid way precisely what features make KdV algebro-geometrically integrable. The introduction warns the reader that the symmetry

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that governs the five chapters makes for some repetition; I have not found that bothersome in the least. Each equation has features that beckon (but are not identical) to the previous, and added ones; it is very useful to have them laid out in parallel. The writing is often humourous and the book interspersed with beautiful literary quotes (in English, French, German and Norwegian) and anecdotes.

To conclude: this is a book that I would recommend to any student of mine, for clarity and completeness of exposition and a "view from the top" of algebraic techniques that generate so many important solutions to equations of mathematical physics. The book provides excellent training even without exercises; the authors made the conscious decision of taking the reader through a detailed analysis of algorithms and techniques rather than stating them as tantalizing exercises, since it's true that finding exercises pertaining to this theory is equivalent to assigning a master's thesis, whereas from reading this book with "pencil and paper" a newcomer can gain a toolkit that enables active research. Any expert as well would enjoy the book and learn something stimulating from the sidenotes that point to alternative developments. We look forward to volumes two and three!

Emma Previato (1-BOST)