Lin & Segel's physiological flow problem

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A brief description of the model

Osmosis is the transport of water through a membrane resulting from differences in the solute concentrations, C_1 vs. C_2, of the two sides of the membrane. The resulting flow rate, in volume per unit area and time unit is given by

$$J = P(C_2 - C_1) \tag{1}$$

Here the units are: $[J] = \mathrm{m}^3 \cdot \mathrm{m}^{-2}\mathrm{s}^{-1} = \mathrm{m\,s}^{-1}$; $[C_i] = \mathrm{osmol\,m}^{-3}$; and hence $[P] = \mathrm{m}^4\mathrm{osmol}^{-1}\mathrm{s}^{-1}$.

Diffusion is given by Fick's law: The solute flux is proportional to the concentration gradient,

$$F_D = -D\frac{\partial C}{\partial x} \tag{2}$$

where units are $[F_D] = \mathrm{osmol\,m}^{-2}\mathrm{s}^{-1}$; and $[D] = \mathrm{m}^2\mathrm{s}^{-1}$. D is called *the diffusion coefficient.*

Chemical pumps at the inner end of the channel move solute into the channel via a *chemical pump* whose mechanism we will not try to understand here. This is postulated to take place at a constant rate N_0 per area and time, that is, $[N_0] = \mathrm{osmol\,m}^{-2}\mathrm{s}^{-1}$.

We model the channel as a long thin channel of constant cross-sectional area a, circumference c, and length L (see figure 1). The chemical pump acts only at the inner end of the channel, in a section of length δ. Since the channel is assumed to be thin, we will assume all quantities to be constant over any cross section of the channel, so that we need only a single space coordinate, namely x^* (with $x^* = 0$ at the innermost end, and $x^* = L$ at the outlet).

We look for a *stationary state* for the system, so that all unknown quantities are functions of x^* alone. Unknowns are the solute concentration C^* and the water velocity v^*.

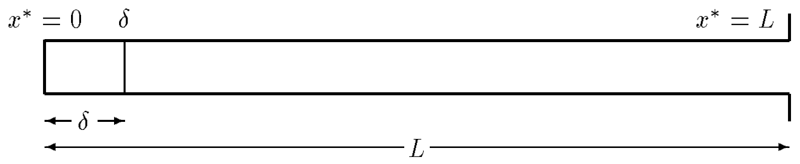

Figure 1: The narrow channel

Another quantity of interest is the solute concentration C_0 in the surrounding tissue, which is also assumed to be the solute concentration at the outlet of the channel.

Mass balance for the water. Since water is neither created nor destroyed, the net amount of water leaving the channel between x_1^* and x_2^* (figure 2) must be zero:

$$av^*(x_2^*) - av^*(x_1^*) - \int_{x_1^*}^{x_2^*} cJ(x^*)dx^* = 0$$

where J is given by (1) (with $C_2 = C^*$ and $C_1 = C_0$). This can be

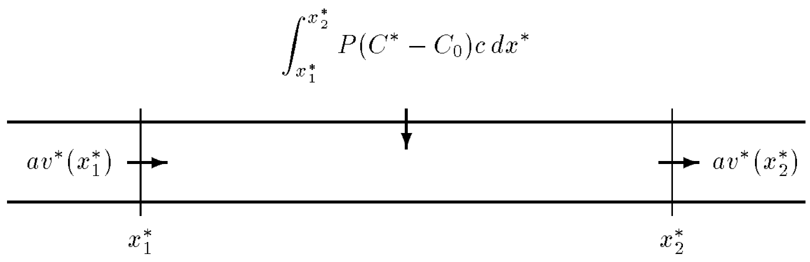

Figure 2: Conservation of water

rewritten using the fundamental theorem of calculus into

$$\int_{x_1^*}^{x_2^*} \left(a\frac{dv^*}{dx^*} - cJ(x^*) \right) dx^* = 0$$

and so the integrand must be zero if this is to hold for every choice of x_1^* and x_2^*:

$$\frac{dv^*}{dx^*} = \frac{c}{a}J(x^*) = \frac{c}{a}P(C^* - C_0) \tag{3}$$

Mass balance for solute. Solute, like water, is neither created nor destroyed in this process. Figure 3 applies the the innermost region $x^* < \delta$; when $x^* > \delta$ on the other hand, the contribution from the term $N_0c(x_2^* - x_1^*)$ should be replaced by 0. F^* represents the total

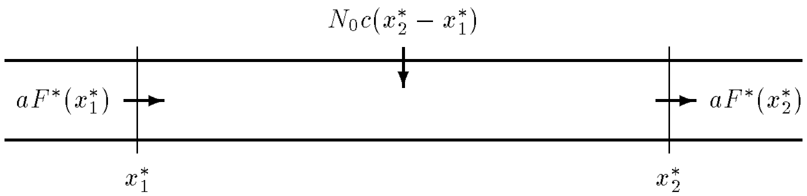

Figure 3: Conservation of solute

solute flux in the channel, and is the sum of a *diffusive* and a *convective* term: $F^* = F_D^* + F_K^*$ where F_D^* is given by (2) and $F_K^* = v^*C^*$. A quick glance at the figure leads to the equation

$$aF^*(x_2^*) - aF^*(x_1^*) - N_0c(x_2^* - x_1^*) = 0, \qquad 0 \le x_1^* < x_2^* < \delta$$

We could of course change this to a differential equation as above, but instead we can use it to find an expression for $F^*(x^*)$ by setting $x_1^* = 0$ and $x_2^* = x^*$. Because $F^*(0) = 0$ we get $aF^*(x^*) = cN_0x^*$. For $\delta \le x_1^* < x_2^* < L$ it is simpler still, since the term $N_0c(x_2^* - x_1^*)$ is dropped, and we conclude that $F^*(x^*)$ is constant for $x^* \ge \delta$. Thus we

find

$$v^*C^* - D\frac{dC^*}{dx^*} = F^*(x^*) = \begin{cases} \frac{c}{a}N_0x^* & \text{for } 0 \le x^* \le \delta \\ \frac{c}{a}N_0\delta & \text{for } \delta \le x^* \le L \end{cases} \tag{4}$$

Boundary conditions. In deriving (4) we already used one boundary condition, namely $F^*(0) = 0$. Since this is automatically fulfilled when (4) holds, we no longer need consider this. Thus we are left with two boundary conditions (5) and (6), plus the *interior* boundary conditions (7):

$$v^*(0) = 0 \tag{5}$$

$$C^*(L) = C_0 \tag{6}$$

$$v^*, C^*, \frac{dC^*}{dx^*} \text{ continuous at } x^* = \delta \tag{7}$$

The inner boundary conditions need to be added because (4) introduces a potential discontinuity into the problem for $x^* = \delta$. The continuity of v^* is necessary for water conservation, and $F^* = v^*C^* - D(dC^*/dx^*)$ must be continuous for the sake of solute conservation. C^* must be continuous because diffusion will make it so, and it follows that dC^*/dx^* is continuous as well.

Dimensional analysis and scaling

We have arrived at a model given by equations (3), (4), (5), (6), (7). The parameters of the problem occur only in the combinations cP/a, cN_0/a, C_0, D, δ and L. which does simplify dimensional analysis somewhat. We start with these parameter combinations and their units:

δ	C_0	$\frac{c}{a}N_0$	$\frac{c}{a}P$	D	L
m	$\frac{\text{osmol}}{\text{m}^3}$	$\frac{\text{osmol}}{\text{m}^3\text{s}}$	$\frac{\text{m}^3}{\text{osmol s}}$	$\frac{\text{m}^2}{\text{s}}$	m

From these six quantities we can, as expected, produce three independent dimensionless combinations, for example

$$\nu = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0D}{cN_0\delta^2}, \quad \lambda = \frac{L}{\delta}$$

Scaling. The solute flux must clearly change from 0 to its maximum value in the region from $x^* = 0$ to $x^* = \delta$. This ought to make δ a good length scale. We need a suitable scale for v^*. Unless the diffusion term dominates completely in (4), the other two terms in that equation must be the same order of magnitude, which we write $v^* C^* \sim cN_0\delta/a$. Assuming $C^* \sim C_0$ we then get $v^* \sim cN_0\delta/(aC_0)$, which we chose as a scale for v^*. So we scale the problem using

$$\begin{aligned} C^* &= C_0 C \\ v^* &= \frac{cN_0\delta}{aC_0} v \\ x^* &= \delta x \end{aligned} \tag{8}$$

and the complete problem becomes (from now on writing $C' = dC/dx$, etc)

$$\begin{aligned} \nu v' &= C - 1 \\ Cv - \eta C' &= \begin{cases} x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 \le x \le \lambda \end{cases} \\ v(0) &= 0 \\ C(\lambda) &= 1 \\ v, C, C' &\text{ continuous at } x = 1 \end{aligned}$$

Perturbation analysis

A somewhat naïve attempt at a perturbation analysis is to treat ν as a small parameter, and to express v and C as power series in ν. If we put $v = v_0 + \nu v_1 + \ldots$ and $C = G_0 + \nu G_1 + \ldots$,[1] substitute these in the model, and compute to second order in ν we find

$$\nu v_0' = G_0 - 1 + \nu G_1 + \mathcal{O}(\nu^2) \implies G_0 = 1, v_0' = G_1$$

[1] The symbol C_0 is already in use, so I use G_i for the coefficients of the power series for C. The alternative is Lin & Segel's $C^{(i)}$, which I find overly complex.

and

$$G_0 v_0 + \nu(G_1 v_0 + G_0 v_1) - \eta G_0' - \nu\eta G_1' = \begin{cases} x + \mathcal{O}(\nu^2) & \text{for } 0 \le x \le 1 \\ 1 + \mathcal{O}(\nu^2) & \text{for } 1 \le x \le \lambda \end{cases}$$

from which $v_0 = x$ for $x < 1$ and $v_0 = 1$ for $x > 1$, and hence $G_1 = 1$ for $x < 1$ and $G_1 = 0$ for $x > 1$. But this contradicts the requirement of continuity of C at $x = 1$!

A more successful attempt. From a study of the numerical values of the parameters we see that N_0 is the most variable of all the quantities in the problem, ranging over several orders of magnitude. If we change N_0 so that ν becomes small we see that η has a tendency to become large at the same time, with $\nu\eta$ approximately constant. It might, therefore, be more useful to assume $\nu\eta$ constant in the perturbation expansion. As it turns out, the final answer becomes a bit simplify if we name this constant as follows:

$$\nu\eta = \psi^{-2}$$

Elimination of η from the problem now converts it into

$$\begin{aligned} \nu\frac{dv}{dx} &= C - 1 \\ \nu C v - \frac{1}{\psi^2}\frac{dC}{dx} &= \nu \begin{cases} x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 \le x \le \lambda \end{cases} \\ v(0) &= 0 \\ C(\lambda) &= 1 \\ v, C, C' &\text{ continuous at } x = 1 \end{aligned}$$

Again we put $v = v_0 + \nu v_1 + \cdots$ and $C = G_0 + \nu G_1 + \cdots$, and find

$$\begin{aligned} \nu v_0' &= G_0 - 1 + \nu G_1 + \mathcal{O}(\nu^2) \\ \nu G_0 v_0 - \psi^{-2} G_0' - \nu\psi^{-2} G_1' &= \nu \begin{Bmatrix} x \\ 1 \end{Bmatrix} + \mathcal{O}(\nu^2) \\ v_0(0) + \nu v_1(0) + \cdots &= 0 \\ G_0(\lambda) + \nu G_1(\lambda) + \cdots &= 1 \end{aligned}$$

where the expression in braces is short for "pick the upper expression for $x < 1$, and the lower one for $x > 1$." For the constant term, we find the two equations $G_0 - 1 = 0$ and $-\psi^{-2}G_0' = 0$, which fortunately are consistent, with the solution $G_0 = 1$. This also fits the boundary condition $G_0(\lambda) = 1$. For the first order term (coefficient of ν^1) we get

$$v_0' = G_1, \quad v_0 C_0 - \psi^{-2} C_1' = \begin{Bmatrix} x \\ 1 \end{Bmatrix}$$

with the solution

$$v_0 - \psi^{-2} v_0'' = \begin{Bmatrix} x \\ 1 \end{Bmatrix}$$

The general solution in $0 < x < 1$ is $v_0 = x + Ae^{\psi x} + Be^{-\psi x}$, but from $v_0(0) = 0$ we then get

$$v_0(x) = x - K_1 \sinh(\psi x), \qquad 0 < x < 1$$

The general solution in $1 < x < \lambda$ is $v_0 = 1 + Ae^{\psi x} + Be^{-\psi x}$, but since $v_0' = G_1$ and $G_1(\lambda) = 0$ we find $v_0'(\lambda) = 0$, and we can write

$$v_0(x) = 1 - K_2 \cosh(\psi(x - \lambda)), \qquad 1 < x < \lambda$$

Since $v_0' = G_1$ and C is continuous at $x = 1$, we see that not only is v_0 continuous at $x = 1$, but so is v_0'. This produces two equations for the determination of K_1 and K_2:

$$\begin{aligned} 1 - K_1 \sinh(\psi) &= 1 - K_2 \cosh(\psi(1 - \lambda)) \\ 1 - K_1 \psi \cosh(\psi) &= -K_2 \psi \sinh(\psi(1 - \lambda)) \end{aligned}$$

Rewrite this in matrix form and use the identity $\cosh(\alpha - \beta) = \cosh(\alpha)\cosh(\beta) - \sinh(\alpha)\sinh(\beta)$ to discover the matrix has determinant $\cosh(\psi\lambda)$, from which we quickly find the solution

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \frac{1}{\psi \cosh(\psi\lambda)} \begin{pmatrix} \cosh(\psi(1 - \lambda)) \\ \sinh(\psi) \end{pmatrix}$$

using Cramer's rule. If you wish to compare this with the results of Lin & Segel, you should note that their κ is my $\psi\lambda$.

We are ultimately interested in Os^*, which is defined as the total solute flux at the outlet divided by the total water flux there. Going back to equation (4), we see that $F^*(L) = (c/a)N_0\delta$, so that

$$\mathrm{Os}^* = \frac{cN_0\delta}{av^*(L)} = \frac{C_0}{v(\lambda)}$$

where we used (8). This might be a good point at which to introduce dimensionless osmolarity

$$\mathrm{Os} = \frac{\mathrm{Os}^*}{C_0} = \frac{1}{v(\lambda)} = \frac{1}{1 - K_2} + \mathcal{O}(\nu)$$

In many cases of interest, ψ is relatively small, so that $\sinh(\psi) \approx \psi$. If this is introduced into the expression for K_2 we get $K_2 \approx 1/\cosh(\psi\lambda)$, so that

$$\mathrm{Os} \approx \frac{\cosh(\kappa)}{\cosh(\kappa) - 1}$$

where $\kappa = \psi\lambda$. For a discussion of these results and a comparison with numerical and experimental results I refer to Lin & Segel.

Reference

C.C Lin & L.A. Segel: *Mathematics applied to deterministic problems in the natural sciences.* ISBN 0898712297. Chapter 8 (pp. 244–276).