The uniform boundedness theorem

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The purpose of this note is to present an alternative proof of the uniform boundedness theorem, without the need for the Baire category theorem.

I found this proof in Emmanuele DiBenedetto: Real Analysis. DiBenedetto refers to an article by W. F. Osgood: Nonuniform convergence and the integration of series term by term, Amer. J. Math., 19, 155–190 (1897). Indeed, the basic idea of the following proof seems to be present in that paper, although the setting considered there is much less general: It is concerned with sequences of functions on a real interval.

I have rewritten the proof a bit, splitting off the hardest bit into a lemma.

1 Lemma. Let \((X, d)\) be a complete, nonempty, metric space, and let \(F\) be a set of real, continuous functions on \(X\). Assume that \(F\) is pointwise bounded from above, in the following sense: For any \(x \in X\) there is some \(c \in \mathbb{R}\) so that \(f(x) \leq c\) for all \(f \in F\). Then \(F\) is uniformly bounded from above on some nonempty open subset \(V \subseteq X\), in the sense that there is some \(M \in \mathbb{R}\) so that \(f(x) \leq M\) for all \(f \in F\) and all \(x \in V\).

Proof: Assume, on the contrary, that no such open subset exists.

That is, for every nonempty open subset \(V \subseteq X\) and every \(M \in \mathbb{R}\), there exists some \(f \in F\) and \(x \in V\) with \(f(x) > M\).

In particular (starting with \(V = X\)), there exists some \(f_1 \in F\) and \(x_1 \in X\) with \(f_1(x_1) > 1\). Because \(f_1\) is continuous, there exists some \(\varepsilon_1 > 0\) so that \(f_1(z) \geq 1\) for all \(z \in B_{\varepsilon_1}(x_1)\).

We proceed by induction. For \(k = 2, 3, \ldots\), find some \(f_k \in F\) and \(x_k \in B_{\varepsilon_{k-1}}(x_{k-1})\) so that \(f_k(x_k) > k\). Again, since \(f_k\) is continuous, we can find some \(\varepsilon_k > 0\) so that \(f_k(z) \geq k\) for all \(z \in B_{\varepsilon_k}(x_k)\). In addition, we require that \(B_{\varepsilon_k}(x_k) \subseteq B_{\varepsilon_{k-1}}(x_{k-1})\), and also \(\varepsilon_k < k^{-1}\).

Now we have a descending sequence of nonempty closed subsets

\[
X \supseteq B_{\varepsilon_1}(x_1) \supseteq B_{\varepsilon_2}(x_2) \supseteq B_{\varepsilon_3}(x_3) \supseteq \cdots,
\]

and the diameter of \(B_{\varepsilon_k}(x_k)\) converges to zero as \(k \to \infty\). Since \(X\) is complete, the intersection \(\bigcap_k B_{\varepsilon_k}(x_k)\) is nonempty; in fact, \((x_k)_k\) is a Cauchy sequence converging to the single element \(x\) of this intersection.

But now \(f_k(x) \geq k\) for every \(k\), because \(x \in B_{\varepsilon_k}(x_k)\). However that contradicts the upper boundedness of \(F\) at \(x\), and this contradiction completes the proof. \(\blacksquare\)
2 Theorem. (Uniform boundedness) Let $X$ be a Banach space and $Y$ a normed space. Let $\Phi \subseteq B(X, Y)$ be a set of bounded operators from $X$ to $Y$ which is point-wise bounded, in the sense that, for each $x \in X$ there is some $c \in \mathbb{R}$ so that $\|Tx\| \leq c$ for all $T \in \Phi$. Then $\Phi$ is uniformly bounded: There is some constant $C$ with $\|T\| \leq C$ for all $T \in \Phi$.

**Proof:** Apply Lemma 1 to the set of functions $x \mapsto \|Tx\|$ where $T \in \Phi$. Thus, there is an open set $V \subseteq X$ and a constant $C$ so that $\|Tx\| \leq C$ for all $T \in \Phi$ and all $x \in V$.

Pick some $z \in V$ and $\varepsilon > 0$ so that $B_\varepsilon(z) \subseteq V$. Also fix $c \in \mathbb{R}$ with $\|Tx\| \leq c$ whenever $T \in \Phi$. Now, if $\|x\| \leq 1$ then $z + \varepsilon x \in V$, and so for any $T \in \Phi$ we get

$$\|Tx\| = \|\varepsilon^{-1}(T(z + \varepsilon x) - Tz)\| \leq \varepsilon^{-1}(\|T(z + \varepsilon x)\| + \|Tz\|) \leq \varepsilon^{-1}(M + c).$$

Thus $\|T\| \leq \varepsilon^{-1}(M + c)$ for any $T \in \Phi$. 

\[ \blacksquare \]