Assorted notes on complex function theory

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Abstract. These are supplementary notes for a course on complex function theory. The notes were first made for the course in 2006. For 2007, those notes were worked into a single document and some more material has been added.

The basic text for the course is Nakhlé H. Asmar's *Applied complex analysis with partial differential equations*. These notes are only intended to fill in some material that is not in Asmar's book, or to present a different exposition.

The course also has a section on Fourier series and separation of variables. This explains the curious presence of a proof of the Riemann–Lebesgue lemma at the end.

The present version is an update made after the fall term of 2007 was over. While teaching from the notes, I noticed some missing bits and found some slightly better proofs, and this version incorporates some of these.

The writing is probably still too terse for many students who are not used to reading mathematics in this form, so there is plenty of room for improvement.

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Chapter 1

A small mouthful of topology

The real numbers

A fundamental property of the real numbers, that separates them from the rational numbers, is the *completeness axiom*. It has several equivalent forms. Here we shall concentrate on one of them. But first a few definitions. We write $\mathbb{R}$ for the set of real numbers.

1 Definition. A set of real numbers $A \subseteq \mathbb{R}$ is called *bounded above* if there exists a real number M so that $a \le M$ for all $a \in A$. Such a number is called an *upper bound* for A. We also say that A is *upward bounded,* since "bounded above" can be grammatically awkward sometimes.

Similarly, A is called *bounded below* (or *downward bounded*) if there exists a real number m so that $a \ge m$ for all $a \in A$. Such a number is called a *lower bound* for A.

When A er bounded both above and below, we simply call it *bounded.* An equivalent formulation is that there is some real number M so that $|a| \le M$ for all $a \in A$. This definition makes sense for sets of *complex* numbers too, and we shall adopt this definition in that case.

The completeness axiom for the real numbers states that if A is a nonempty set of real numbers and A is bounded above, then A has a *least upper bound,* that is an upper bound S with the property that any other upper bound M satisfies $S \le M$. This least upper bound is also called the *supremum* of A, and is written $S = \sup A$.

If A has no upper bound, we write $\sup A = +\infty$. To this we add the definition $\sup \emptyset = -\infty$, so that every subset of $\mathbb{R}$ has a supremum.

Note that $\sup A$ is uniquely determined by the requirements

$$a \le \sup A \text{ for all } a \in A, \text{ and}$$
$$\text{for every } b < \sup A \text{ there is some } a \in A \text{ with } a > b.$$

It follows from the completeness axiom that every nonempty set that is bounded below has a greatest lower bound, which we call the *infimum* of the set and write $\inf A$. (Exercise: Show this! Hint: You should have $\inf A = \sup(-A)$, where $-A = \{a\colon -a \in A\}$.)

Below I write a sequence of numbers $(x_1, x_2, x_3, \ldots)$ as $(x_k)_{k=1}^{\infty}$, or yet more briefly as (x_k). The sequence is called *increasing* if $x_{k+1} \ge x_k$ for all k, and *decreasing* if $x_{k+1} \le x_k$ for all k.[1] It is called *monotone* if it is either decreasing or increasing.

2 Proposition. *Every monotone, bounded sequence of real numbers is convergent.*

Proof: I show this only for a increasing, bounded sequence (x_k). The proof for decreasing sequences is similar, or it follows from the increasing case by replacing x_k with $-x_k$.

Let $s = \sup\{x_k\colon k = 1, 2, 3, \ldots\}$. I claim that $x_k \to s$ when $k \to \infty$.

To see this, let $\varepsilon > 0$. Then there exists some n with $x_n > s - \varepsilon$. Then $k \ge n$ for all $s - \varepsilon < x_n \le x_k \le s$, and from this we obtain $|x_k - s| < \varepsilon$. We have shown that $x_k \to s$, as claimed. ∎

3 Lemma. *If A if a nonempty, closed, upward bounded set of real numbers then* $\sup A \in A$.

Proof: Let $s = \sup A$. For every $\varepsilon > 0$ there is some $x \in A$ with $s - \varepsilon < x \le s$. Then $|s - x| < \varepsilon$, which shows that $s \in A$ since A is closed. ∎

4 Theorem. *Assume that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a decreasing sequence of nonempty closed and bounded subsets of* $\mathbb{R}$. *Then all these sets have at least one point in common; that is, the intersection $K_1 \cap K_2 \cap K_3 \cap \ldots$ is nonempty.*

More briefly put: A decreasing sequence of nonempty, closed subsets of $\mathbb{R}$ has a nonempty intersection.

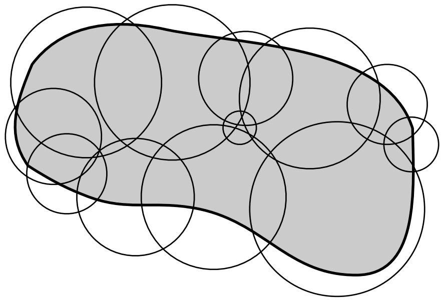
Proof: Let $s_n = \sup K_n$ for $n = 1, 2, 3, \ldots$. Then $s_1 \ge s_2 \ge s_3 \ge \cdots$, and (s_n) is a bounded sequence since it is contained in the bounded set K_1, so it has a limit, say $s_n \to s$ when $n \to \infty$. Since $s_k \in K_k \subseteq K_n$ when $k \ge n$, and since K_n is closed, we must have $s \in K_n$. ∎

[1] If you replace the inequalities in this definition by strict inequalities, we talk of *strictly* increasing or decreasing sequences. In an unfortunate twist of terminology, “increasing” has come to mean “strictly increasing”, and similarly with “decreasing”, forcing the use of the horrible terminology “nondecreasing” and “nonincreasing” for the non-strict versions of these concepts. It has gotten so bad that many authors only use terms like “strictly increasing” or “nondecreasing” out of fear that “increasing” will be misunderstood. Beware of this mess when reading the literature.

Complex numbers and compactness

5 Definition. Consider a subset $K \subseteq \mathbb{C}$ of the complex numbers. An *open cover* of K is a set $\mathcal{U}$ consisting of open subsets of $\mathbb{C}$ whose union contains K. Briefly: $\bigcup \mathcal{U} \supseteq K$. We also say that $\mathcal{U}$ *covers* K when $\bigcup \mathcal{U} \supseteq K$.

K is called *compact* if it has the following property: In any open cover $\mathcal{U}$ of K one can find a finite number of members that together cover K.

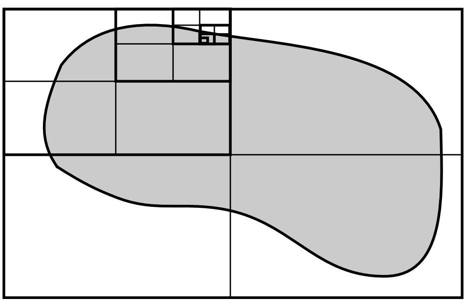


A set covered by a finite number of circles.

6 Theorem. (Heine–Borel) *A set of complex numbers is compact if and only if it is closed and bounded.*

Proof: We start with the most interesting and useful part: Assume that K is closed and bounded. We shall prove that K is compact.

Assume that K is not compact; we shall derive a contradiction. Assume therefore that $\mathcal{U}$ is an open cover of K so that no finite set of members of $\mathcal{U}$ covers K.



A set with zooming rectangles $R_0 \supset R_1 \supset \cdots$.

Since K is bounded, the set is contained in some closed rectangle R_0 with sides parallel to the coordinate axes. We divide R_0 along the middle in the horizontal and vertical directions, into four closed rectangles $R_{0,1}$, $R_{0,2}$, $R_{0,3}$ and $R_{0,4}$. Since

$$K = (K\cap R_{0,1}) \cup (K\cap R_{0,2}) \cup (K\cap R_{0,3}) \cup (K\cap R_{0,4})$$

it must be impossible to cover at least one of the four sets $K\cap R_{0,j}$ using a finite number of members of $\mathcal{U}$. Write $R_1 = R_{0,j}$ for one such j. Now divide R_1 in the same way, and continue the process forever to produce rectangles $R_0 \supset R_1 \supset R_2 \supset \cdots$ where no $K\cap R_k$ can be covered by a finite number of members of $\mathcal{U}$.

We show first that all the rectangles R_k have a point in common. Write $R_k = \{x+iy\colon x\in[a_k,b_k], y\in[c_k,d_k]\}$, where $[a_0,b_0]\supset[a_1,b_1]\supset\cdots$ and $[c_0,d_0]\supset[c_1,d_1]\supset\cdots$. By Theorem 4 all the intervals $[a_k,b_k]$ have a common point x, all the intervals $[c_k,d_k]$ have a common point y, and the point $z=x+iy$ will then be common to all the R_k.

Next, pick a point in $z_k\in K\cap R_k$ for $k=0,1,2,\ldots$. Since the length of the sides of R_k go (exponentially even) to zero as $k\to\infty$ and also $z\in R_k$, we must have $z_k\to z$. Because $z_k\in K$ and K is closed, we conclude that $z\in K$.

Since $\mathcal{U}$ covers K, there is some $U\in\mathcal{U}$ with $z\in U$. But again, since the sides of R_k decrease towards zero and U is open, we find that $R_k\subseteq U$ when k is large enough. But then R_k is covered by a single set from $\mathcal{U}$, which definitely contradicts the construction of R_k as a set that cannot be covered by any finite set of members from $\mathcal{U}$. This contradiction completes this half of the proof.

For the opposite implication, assume that K is compact. To show that K is bounded, just note that all the balls $B_r(0)$ where $r>0$ form an open cover for K. Since a finite number of these will cover K, then indeed the largest of this finite number of balls covers K, so K is bounded.

To show that K is closed, pick any point $z\notin K$. Then the exteriors of all balls centered at z, $\{w\in\mathbb{C}\colon |w-z|>\delta\}$ (with $\delta>0$), form an open cover of K. Again, a finite number of them will cover K, and then the largest of them (corresponding to the smallest δ) will cover K. Then $K\cap B_\delta(z)=\emptyset$, and we're done. ∎

We can now generalize Theorem 4 to the complex numbers. And not only that, but we can move beyond decreasing sequences of sets:

7 Theorem. *Assume that $\mathcal{F}$ is a set of compact subsets of $\mathbb{C}$. Assume further that $F_1\cap\cdots\cap F_n\neq\emptyset$ whenever $F_1,\ldots,F_n\in\mathcal{F}$. Then all the sets $\mathcal{F}$ have at least one common point.*

The second assumption is sometimes called the *finite intersection property* for the set $\mathcal{F}$.

Proof: Let F_0 be some fixed member of $\mathcal{F}$. Let $\mathcal{U} = \{\mathbb{C} \setminus F \colon F \in \mathcal{F}\}$. If the sets in $\mathcal{F}$ have no point in common, then $\mathcal{U}$ is an open cover of F_0 (in fact, of $\mathbb{C}$). Since F_0 is compact, a finite number of sets from $\mathcal{U}$ will cover F_0. But then

$$F_0 \subseteq (\mathbb{C} \setminus F_1) \cup \cdots \cup (\mathbb{C} \setminus F_n) = \mathbb{C} \setminus (F_1 \cap \cdots \cap F_n),$$

and $F_0 \cap F_1 \cap \cdots \cap F_n = \emptyset$. This contradicts our assumption on $\mathcal{F}$. ∎

Here is a rather immediate consequence:

8 Corollary. *A real, continuous function defined on a nonempty, compact set achieves its maximum and minimum on the set.*

In other words, if K is compact and f is a real, continuous function defined on K then there are $a, b \in K$ so that $f(a) \le f(z) \le f(b)$ for all $z \in K$. In particular, f is a bounded function.

Proof: With the notation as above we show only the existence of b. The existence of a has a similar proof, or it follows from the existence of b applied to the function $-f$.

For every real number t define the set $F_t = \{z \in K \colon f(z) \ge t\}$. Since K is closed and f is continuous, F_t is also closed. Since it is contained in the compact set K it is also bounded, so it is in fact compact.

If $F_n \ne \emptyset$ for all natural numbers n, we can use Theorem 7 and find some common point z of all the sets F_n. That means that $f(z) \ge n$ for all n, which is absurd. Therefore $F_n = \emptyset$ for at least one n, so f is bounded above.

Now let $s = \sup\{f(z) \colon z \in K\} < \infty$, and apply the previous argument to all the sets F_t with $t < s$. All these sets are nonempty thanks to the definition of s, so they have a point in common thanks to Theorem 7. If $b \in F_t$ for all $t < s$ then $f(b) = s$, and we're done. ∎

An application. Let $\Omega \subset \mathbb{C}$ be a proper, open subset of $\mathbb{C}$. We can define the *distance to the complement* as

$$d(z) = \inf\{|z - w| \colon w \in \mathbb{C} \setminus \Omega\}, \qquad z \in \mathbb{C}.$$

Since Ω is open $d(z) > 0$ when $z \in \Omega$. Yet more obvious is the fact that $d(z) = 0$ when $z \in \mathbb{C} \setminus \Omega$ (pick $w = z$ to see this).

I claim that d *is continuous*: For if $z, \zeta \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \Omega$ then $|z - w| = |z - \zeta + \zeta - w| \le |z - \zeta| + |\zeta - w|$. If we take the infimum over all $w \in \mathbb{C} \setminus \Omega$ we conclude $d(z) \le |z - \zeta| + d(\zeta)$. Rewrite that as $d(z) - d(\zeta) \le |z - \zeta|$. If we interchange z and ζ

we also get $d(\zeta) - d(z) \le |z - \zeta|$, and so $|d(z) - d(\zeta)| \le |z - \zeta|$. The continuity of d follows immediately from this inequality. [2]

Now let K be a compact subset of Ω. Then the continuous function d must achieve its minimum on K. So there exists some $a \in K$ so that $d(a) \le d(z)$ for all $z \in K$. But then $d(a) > 0$ since $a \in K \subseteq \Omega$. This smallest distance $d(a)$ is simply called the *distance* from K to $\mathbb{C} \setminus \Omega$, and we have shown that this distance is positive.[3]

9 Definition. A number sequence (z_k) is called a *Cauchy sequence* if, for every $\varepsilon > 0$, there is some natural number n so that each time $j \ge n$ and $k \ge n$ then $|z_j - z_k| < \varepsilon$.

A bit more carelessly stated: Any two numbers sufficiently far out in the sequence will be arbitrarily close together. We often use the word "Cauchy" as an adjective, saying about a sequence that is is Cauchy.

It is not hard to prove that any convergent sequence is Cauchy. The converse is also true:

10 Theorem. *Every Cauchy sequence is convergent.*

Proof: Let (z_n) be a Cauchy sequence. We consider the *tail* $T_n = \{z_k : k \ge n\}$ for each n. It is not hard to prove that the sequence (z_n) is bounded, so each tail T_n is itself a bounded set. Its closure $\overline{T_n}$ is therefore compact. Since $\overline{T_1} \supseteq \overline{T_2} \supset \cdots$, all the sets $\overline{T_n}$ have a common point z. I claim that $z_n \to z$.

Let $\varepsilon > 0$. Pick n as in Definition 9. Since $z \in \overline{T_n}$ there is, according to the definition of the closure of some member of T_n – that is, some z_j with $j \ge n$ – with $|z - z_j| < \varepsilon$. For every $k \ge n$ we then find $|z - z_k| \le |z - z_j| + |z_j - z_k| < 2\varepsilon$, and $z_n \to z$ follows from this. ∎

Uniform continuity. Uniform continuity is a somewhat technical device that is sometimes needed for the sake of getting estimates, particularly in connection with integration.

11 Definition. A function f defined on a set A is called *uniformly continuous* on A if for each $\varepsilon > 0$ there exists some $\delta > 0$ so that for all $x, y \in A$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

[2] More generally, a function f is called *Lipschitz continuous* (some times we say only Lipschitz) if there is a constant L so that $|f(z) - f(w)| \le L|z - w|$ for all z and w. The smallest such L is called the *Lipschitz constant* of f. Such a function is also uniformly continuous (to be defined later): In the definition you may pick $\delta = \varepsilon / L$.

[3] We often prefer to say "the distance to the boundary" rather than "the distance to the complement". The two distances are actually the same. (Prove this!)

The difference between uniform continuity and ordinary (pointwise) continuity is that in uniform continuity the same δ can be used for a given ε everywhere in the space. For pointwise continuity we must allow δ to depend not only on ε, but also on x.

12 Example. Consider the function $f(x) = x^2$. Then $|f(x) - f(y)| = |x+y||x-y|$, which shows that f is not uniformly continuous on $\mathbb{R}$: For given $\delta > 0$ and $\varepsilon > 0$ we can always pick x and y so that $|x-y| = \delta$ and $|x+y| > \varepsilon/\delta$, which gives $|f(x) - f(y)| > \varepsilon$.

In a similar way we can show that the function $g(x) = 1/x$ is not uniformly continuous on $(0,\infty)$. (Use for example $|g(x) - g(y)| = |x-y|/(xy)$.)

It may seem like this is related to the fact that these functions have very steep graphs as $x \to \infty$ and $x \to 0$ respectively, but that is only a part of the story, for the function $h(x) = \sqrt{x}$ is uniformly continuous $[0,1]$ even though its graph is infinitely steep at $x = 0$.

13 Theorem. *A continuous function is uniformly continuous on every compact subset of its domain.*

Proof: Let f be continuous (real or complex) on a compact set K. Let $\varepsilon > 0$ be any positive number.

Let $\mathcal{U}$ consist of all open balls $B_\delta(z)$ where $z \in K$ and $\delta > 0$ is chosen so that whenever $|w - z| < 2\delta$ then $|f(w) - f(z)| < \varepsilon$.[4] Since f is continuous on all of K, $\mathcal{U}$ will cover K. By compactness therefore, a finite number of them, say $B_{\delta_j}(z_j)$ for $j = 1,2,\ldots,n$ will also cover K. Let $\delta = \min\{\delta_1,\ldots,\delta_n\}$.

Now assume that $w_1, w_2 \in K$ with $|w_1 - w_2| < \delta$. Then $w_1 \in B_{\delta_j}(z_j)$ for a suitable j, so $|w_1 - z_j| < \delta_j$. The triangle inequality also yields $|w_2 - z_j| < 2\delta_j$, so $|f(w_k) - f(z_j)| < \varepsilon$ for $k = 1,2$. A new application of the triangle inequality yields $|f(w_1) - f(w_2)| < 2\varepsilon$. This shows the uniform continuity. ∎

Continuous images of compact sets. We often use continuous functions to map sets to new sets, and then the next result is useful. If f is a function and K is contained in the domain of f then we write $f[K] = \{f(x) : x \in K\}$, and call $f[K]$ *the image of K by f*. We also say that f *maps K onto $f[K]$*.

14 Proposition. *A continuous function maps a compact subset of its domain onto a compact set.*

In other words, if f is continuous and K is compact and contained in the domain of f then $f[K]$ is compact.

[4] The 2δ trick is absolutely necessary here. Note also that δ may depend on z. It is only after we have shown this theorem that we can know that δ could in fact be picked independently of z.

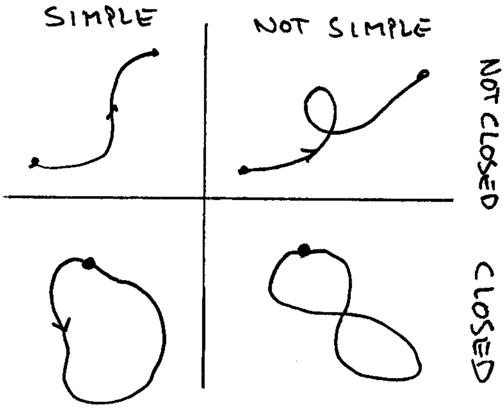
Proof: Let $\mathcal{U}$ be an open cover of $f[K]$. For every $U \in \mathcal{U}$, consider $f^{-1}[U] = \{z \in K : f(z) \in U\}$. Since f is continuous and U is open, $f^{-1}[U]$ is also open. Clearly $\{f^{-1}[U] : U \in \mathcal{U}\}$ covers K, so a finite number of these sets cover K. If $f^{-1}[U_1], \dots, f^{-1}[U_n]$ cover K then clearly $U_1, \dots, U_n$ cover $f[K]$, so $f[K]$ is compact. ∎

Chapter 2

The Cauchy integral theorem

Curves and paths

A (parametrized) *curve* in the complex plane is a continuous map γ from a compact interval $[a,b]$ into $\mathbb{C}$. We call the curve *closed* if its starting point and endpoint coincide, that is if $\gamma(a) = \gamma(b)$. We call it *simple* if it does not cross itself, that is if $\gamma(s) \neq \gamma(t)$ when $s \neq t$. Exception: We allow the curve to be closed, so a better way to say it is that $\gamma(s) = \gamma(t)$ and $s < t$ imply $s = a$ and $t = b$.



A simple, closed curve is often called a *Jordan* curve, because it was Camille Jordan (1838–1922) who first realized that the seemingly obvious fact that such a curve divides the plane into two components – an inside and an outside – was far from obvious, and needed a proof.

Curves are in general quite nontrivial objects. Giuseppe Peano (1858–1932) discovered a curve that covers an entire square in the plane, and William Osgood (1864–1943) found that even a simple curve can have a positive area! (Though it cannot fill a square.)

General curves are to too general for our purpose, which is to use them as integration paths.

Definition of the path integral

Recall the definition of the *Riemann integral*, as the limit of what is known as Riemann sums:

$$\int_a^b f(x)\,dx = \lim \sum_{j=1}^n f(x_j^*)(x_j - x_{j-1})$$

where the sum involves a *partition* of $[a,b]$: That is, a set of points $a = x_0 < x_1 < \cdots < x_n = b$. There are also arbitrary points $x_j^* \in [x_{j-1}, x_j]$. Finally, the limit is to be taken as the partition gets finer and finer, which we may take to mean that its *mesh size*, which is just the maximal value of $x_j - x_{j-1}$, goes to zero.

A classic existence theorem on the Riemann integral states that it exists (which means the limit exists) whenever f is continuous on $[a,b]$.[1]

The Riemann integral as defined here works just as well if f is a complex valued function. If you wish, you can integrate the real part and imaginary parts separately and combine the results, but the definition and all the rules of calculating with it works just fine as they are, even in the complex case.

Since we shall use the path integral as a tool to discover interesting things about the Riemann integral, it is an absolute requirement that the path integral *generalizes* the Riemann integral. We just wish to replace $[a,b]$ by a curve. In some sense, $[a,b]$ *is* a curve, parametrized by itself: Just put $\gamma(t) = t$ for $t \in [a,b]$. To spare the suspense, here is the definition of the path integral.

15 Definition. Assume $\gamma\colon [a,b] \to \mathbb{C}$ is a curve, and f is a function defined on the curve, by which we just mean that whenever $z = \gamma(t)$ then $f(z)$ is defined. Then

$$\int_\gamma f(z)\,dz = \lim \sum_{j=1}^n f(z_j^*)(z_j - z_{j-1}) \tag{2.1}$$

where the points $z_0, z_1, \ldots, z_n$ are points in order along the curve, with the z_j^* in between – or more precisely, we start with a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$, pick $t_j^* \in [t_{j-1}, t_j]$ and put $z_j = \gamma(t_j)$ and $z_j^* = \gamma(t_j^*)$. Then the limit is taken as the partition gets finer, just as in the definition of the Riemann integral. We say the integral *exists* if the limit exists.

Notice that the details of the parametrization plays a very minor role here: It is only used to keep track of, and using, the points along the curve in some prescribed order. Therefore it is immediately obvious that the path integral is independent of the particular parametrization. So, if $[c,d]$ is another interval and $h\colon [c,d] \to [a,b]$ is a continuous, strictly increasing function with $h(c) = a$ and

[1] More precisely, it is known to exist if and only if f is bounded, and the set of points where it is discontinuous has measure zero. Whatever *that* means – I will not define it here.

$h(d) = b$, then we can put $\gamma_1(s) = \gamma(h(s))$ and think of γ_1 as a reparametrization of γ. It follows directly from the definition that $\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz$.

Arc length

Under what circumstances can we expect the arc integral to exist? At the very least, we must have some reassurance that the sums used to define it don't go to infinity as the partition becomes finer. So first, we must assume that the integrand $f(z)$ is bounded along γ, say $|f(\gamma(t))| \le M$ for some finite number M. Even so, the best upper estimate we can think of is

$$\Bigl|\sum_{j=1}^{n} f(z_j^*)(z_j - z_{j-1})\Bigr| \le M \sum_{k=1}^{n} |z_j - z_{j-1}| \le M\ell(\gamma)$$

where $\ell(\gamma)$ is the length of the curve.

That is, if we believe that such a thing as the length of a curve can be defined, and if we further believe that a straight line is the shortest distance between two points: For surely then $|z_j - z_{j-1}|$ is no greater than the length of the curve γ between the two points, and adding up this inequality for $j = 1, 2, \dots, n$ we get the inequality above.

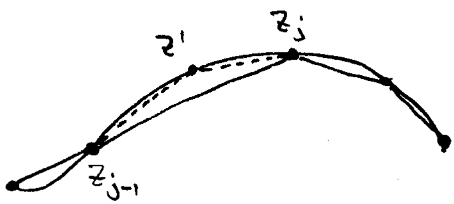
Basically, we define the length of a curve precisely so that this is so.

16 Definition. The *length* of a curve $\gamma\colon [a,b] \to \mathbb{C}$ is

$$\ell(\gamma) = \sup \sum_{j=1}^{n} |z_j - z_{j-1}|, \qquad z_j = \gamma(t_j)$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$. (More precisely, it is the supremum of the set of numbers obtained from the above sum as we consider every partition.)

The curve γ is said to have *finite length* if $\ell(\gamma) < \infty$. In that case, we shall call the curve a *path*.



If we add new points to a partition, we obtain a new partition which is said to be a *refinement* of the original. Clearly, any refinement can be obtained by adding just one point at the time. Now, if we start with a partition $a = t_0 < t_1 < \cdots < t_n = b$ and add to it a single point t' between t_{j-1} and t_j, then one term $z_j - z_{j-1}$ in the sum above will be replaced by two terms, namely

$$|z' - z_{j-1}| + |z_j - z'| \geq |z_j - z_{j-1}|$$

so that the whole sum becomes larger (or at least not smaller). We have shown that the sum in the definition above increases[2] as the partition is refined. Therefore, we could also have defined the length as the *limit* of the above sum as the partition is refined – in analogy with the path integral.

17 Proposition. *If the curve γ is (piecewise) smooth, then*

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Proof: We begin by estimating $|\gamma(t_j) - \gamma(t_{j-1})|$: Clearly

$$\gamma(t_j) - \gamma(t_{j-1}) = \int_{t_{j-1}}^{t_j} \gamma'(t)\,dt.$$

We wish to compare this to $\gamma'(t_j)(t_j - t_{j-1})$:

$$\gamma(t_j) - \gamma(t_{j-1}) - \gamma'(t_j)(t_j - t_{j-1}) = \int_{t_{j-1}}^{t_j} \bigl(\gamma'(t) - \gamma'(t_j)\bigr)\,dt.$$

If γ is smooth then by definition γ' is continuous, and hence uniformly continuous. So given $\varepsilon > 0$, we can choose $\delta > 0$ so that $|t - s| < \delta$ implies $|\gamma(t) - \gamma(s)| < \varepsilon$. If we have chosen a partition with $|t_j - t_{j-1}| < \delta$ for all j, we get $|\gamma'(t) - \gamma'(t_j)| < \varepsilon$ in the above integral, so that

$$|\gamma(t_j) - \gamma(t_{j-1}) - \gamma'(t_j)(t_j - t_{j-1})| \leq (t_j - t_{j-1})\varepsilon$$

and therefore

$$\Bigl||\gamma(t_j) - \gamma(t_{j-1})| - |\gamma'(t_j)|(t_j - t_{j-1})\Bigr| \leq (t_j - t_{j-1})\varepsilon.$$

We may now sum this:

$$\begin{aligned}
&\Bigl|\sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})| - \sum_{j=1}^n |\gamma'(t_j)|(t_j - t_{j-1})\Bigr| \\
&\qquad \leq \sum_{j=1}^n \Bigl||\gamma(t_j) - \gamma(t_{j-1})| - |\gamma'(t_j)|(t_j - t_{j-1})\Bigr| \leq \sum_{j=1}^n (t_j - t_{j-1})\varepsilon = (b-a)\varepsilon.
\end{aligned}$$

[2] At least it does not decrease.

As the partition is refined then we get in the limit

$$\left|\ell(\gamma)-\int_a^b |\gamma'(t)|\,dt\right| \le (b-a)\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we are done. ∎

The existence of the integral

Let us consider what changes happen in the sum as the partition is refined. If $f(\gamma(t))$ varies only slightly in each interval, we expect the sum not to change much with refinement.

So we assume that $\varepsilon>0$ is given, and choose $\delta>0$ so that $|s-t|<\delta$ implies $|f(\gamma(s))-f(\gamma(t))|<\varepsilon$, and we then assume that the partition is chosen so that $0<t_j-t_{j-1}<\delta$.

We concentrate on one term $f\big(\gamma(t_j^*)\big)\big(\gamma(t_j)-\gamma(t_{j-1})\big)$ and what happens when we add new points to the partition, say $t_{j-1}=s_0<s_1<\cdots<s_m=t_j$: Then this term is replaced by a sum, and the difference between the original term and the new sum is

$$\begin{aligned}f\big(\gamma(t_j^*)\big)\big(\gamma(t_j)-\gamma(t_{j-1})\big)-\sum_{k=1}^m f\big(\gamma(s_k^*)\big)\big(\gamma(s_k)-\gamma(s_{k-1})\big)&\\ =\sum_{k=1}^m \Big(f\big(\gamma(t_j^*)\big)-f\big(\gamma(s_k^*)\big)\Big)\big(\gamma(s_k)-\gamma(s_{k-1})\big)&\end{aligned}$$

whose absolute value is not greater than

$$\varepsilon\sum_{k=1}^m |\gamma(s_k)-\gamma(s_{k-1})| \le \varepsilon\ell\big(\gamma|_{[t_{j-1},t_j]}\big)$$

(where $\gamma|_{[t_{j-1},t_j]}$ is that part of γ corresponding to parameter values t lying in $[t_{j-1},t_j]$).

Adding all the terms together, we conclude that when the partition is chosen as explained above, and is then replaced by a refinement, the sum in (2.1) changes by not more than $\varepsilon\ell(\gamma)$.

By a sort of generalizing of Cauchy's convergence criterion (the fact that Cauchy sequences converge) this is enough to guarantee the existence of the integral $\int_\gamma f(z)\,dz$ as a limit of the sum in (2.1) when the partition is refined, and it also yields the estimate

$$\left|\int_\gamma f(z)\,dz-\sum_{j=1}^n f\big(\gamma(t_j^*)\big)\big(\gamma(t_j)-\gamma(t_{j-1})\big)\right| \le \varepsilon\ell(\gamma) \tag{2.2}$$

assuming that the partition is chosen so that $|f(\gamma(s)) - f(\gamma(t))| < \varepsilon$ whenever s and t lie in the same interval $[t_{j-1}, t_j]$ defined by the partition.

To be just a bit more rigorous, write $S(P)$ for the sum in (2.1) associated with a partition P. Let $P_1, P_2, \ldots$ be a sequence of progressively finer partitions, so that the maximal mesh width of P_j goes to zero when $j \to \infty$. The estimates above show that $\big(S(P_j)\big)$ is a Cauchy sequence, and therefore convergent. So we are tempted to define $\int_\gamma f(z)\,dz = \lim_{j\to\infty} S(P_j)$. But what if we had chosen another sequence (P_j^*) of partitions? Could we have obtained a different limit? The answer is no, for P_j and P_j^* have a common refinement P_j^{**}, and $|S(P_j) - S(P_j^*)| \le |S(P_j) - S(P_j^{**})| + |S(P_j^{**}) - S(P_j^*)| \to 0$ when $j \to \infty$, again thanks to the above estimate.

Let us compute some integrals directly from the definition.

18 Lemma. *For every path $\gamma\colon [a,b] \to \mathbb{C}$ we find*

$$\int_\gamma 1\,dz = \gamma(b) - \gamma(a) \quad \textit{and} \quad \int_\gamma z\,dz = \tfrac{1}{2}\big(\gamma(b)^2 - \gamma(a)^2\big).$$

Proof: The first integral is obvious, since *every* approximating sum has the value $\sum_{j=1}^n \big(\gamma(t_i) - \gamma(t_{i-1})\big) = \gamma(t_n) - \gamma(t_0) = \gamma(b) - \gamma(a)$.

The other integral certainly exist, since the integrand is continuous. It can be written as a limit of either of the two sums

$$\sum_{j=1}^n \gamma(t_i)\big(\gamma(t_i) - \gamma(t_{i-1})\big) \quad \text{and} \quad \sum_{j=1}^n \gamma(t_{i-1})\big(\gamma(t_i) - \gamma(t_{i-1})\big),$$

so the integral is also a limit of the mean of the two, that is

$$\frac{1}{2}\sum_{j=1}^n \big(\gamma(t_i) + \gamma(t_{i-1})\big)\big(\gamma(t_i) - \gamma(t_{i-1})\big) = \frac{1}{2}\sum_{j=1}^n \big(\gamma(t_i)^2 - \gamma(t_{i-1})^2\big) = \tfrac{1}{2}\big(\gamma(b)^2 - \gamma(a)^2\big)$$

and the conclusion is once more obvious. ∎

Approximating a path by a broken line

Consider an integration path $\gamma\colon [a,b] \to \mathbb{C}$. Given a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ we can create a new integration path $\gamma^* = [z_0, z_1, \ldots, z_n]$ (where $z_j = \gamma(t_j)$) by joining straight line segments $[\gamma(t_{j-1}), \gamma(t_j)]$: More precisely, we can define $\gamma^*\colon [0,n] \to \mathbb{C}$ by setting

$$\gamma^*(j+s) = s z_j + (1-s) z_{j-1}, \qquad s \in [0,1],\ j = 0, 1, \ldots, n.$$

19 Lemma. *Assume that f is continuous in a neighborhood of an integration path γ. Then with the above notation,*

$$\int_{[z_0,z_1,\dots,z_n]} f(z)\,dz \to \int_\gamma f(z)\,dz$$

with convergence as the partition is refined.

Proof: We find

$$\int_{\gamma^*} f(z)\,dz = \sum_{j=1}^n \int_0^1 f\bigl(sz_j + (1-s)z_{j-1}\bigr)\,ds \cdot (z_j - z_{j-1})$$

which we can compare directly with the sum in (2.1):

$$\begin{aligned}\int_{\gamma^*} f(z)\,dz - \sum_{j=1}^n f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr) \\ = \sum_{j=1}^n \int_0^1 \Bigl(f\bigl(s\gamma(t_j)+(1-s)\gamma(t_{j-1})\bigr) - f\bigl(\gamma(t_j^*)\bigr)\Bigr)\,ds \cdot \bigl(\gamma(t_j)-\gamma(t_{j-1}\bigr)\end{aligned}$$

Here we can ensure that the integrand in the final line has absolute value less than an given $\varepsilon > 0$ by choosing a fine enough partition, and then the absolute value of the entire sum is less than $\varepsilon\ell(\gamma)$. ∎

Piecewise smooth paths

20 Proposition. *If γ is (piecewise) smooth then*

$$\int_\gamma f(z)\,dz = \int_a^b f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt.$$

Proof: We start by comparing one term in (2.1) with the corresponding part of the integral:

$$f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr) - \int_{t_{j-1}}^{t_j} f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt = \int_{t_{j-1}}^{t_j} \Bigl(f\bigl(\gamma(t_j^*)\bigr) - f\bigl(\gamma(t)\bigr)\Bigr)\gamma'(t)\,dt$$

By choosing the partition fine enough we get $|f\bigl(\gamma(t_j^*)\bigr) - f\bigl(\gamma(t)\bigr)| < \varepsilon$, and so we have

$$\left|f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr) - \int_{t_{j-1}}^{t_j} f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt\right| \le \varepsilon \int_{t_{j-1}}^{t_j} |\gamma'(t)|\,dt.$$

Summing this we conclude that

$$\Big|\sum_{j=1}^{n} f\big(\gamma(t_j^*)\big)\big(\gamma(t_j)-\gamma(t_{j-1})\big)-\int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt\Big|\le\varepsilon\int_a^b|\gamma'(t)|\,dt,$$

and the proof is complete. ∎

While the formula above is easier to use than the definition of the integral, the next result is even easier, when it can be used:

21 Proposition. *Assume that f is continuous and has an* antiderivative F, *that is a function so that $F'(z)=f(z)$ for all z in the domain of f. For every path $\gamma\colon[a,b]\to\mathbb{C}$ in the domain of f starting in α and ending in β we then find*

$$\int_\gamma f(z)\,dz=F(\beta)-F(\alpha).$$

Proof: If γ is smooth, then this is obvious, for then

$$\begin{aligned}\int_\gamma f(z)\,dz&=\int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt=\int_a^b\frac{d}{dt}F\big(\gamma(t)\big)\,dt\\&=F\big(\gamma(b)\big)-F\big(\gamma(a)\big)=F(\beta)-F(\alpha).\end{aligned}$$

The result follows for piecewise smooth paths by adding this result over each smooth part.

But then the result follows by approximating a general path γ by broken lines, which are special cases of piecewise smooth paths. ∎

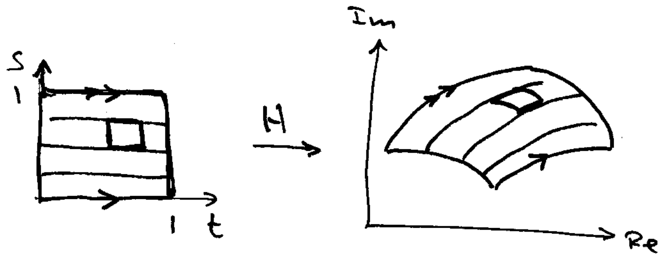
The Cauchy integral theorem

A *homotopy* in a region $\Omega\subseteq\mathbb{C}$ is simply a continuous mapping $H\colon[0,1]\times[0,1]\to\Omega$.

As we keep s fixed, then $t\mapsto H(t,s)$ is a curve in Ω, and similarly if we keep t fixed, then $s\mapsto H(t,s)$ is a curve as well.

So if we consider a small subrectangle $[a,b]\times[c,d]$ of $[0,1]\times[0,1]$ then going around this subrectangle in the positive direction we get a closed path in Ω, which we could parametrize as follows:

$$\gamma(t)=\begin{cases}H\big(tb+(1-t)a,c\big), & 0\le t\le 1,\\ H\big(b,(t-1)d+(2-t)c\big), & 1\le t\le 2,\\ H\big((t-2)a+(3-t)b,d\big), & 2\le t\le 3,\\ H\big(a,(t-3)c+(4-t)d\big), & 3\le t\le 4.\end{cases}$$



In what follows we shall need not only to integrate along these paths, so they have to have finite length, but we need strong estimates on these lengths.

The easiest way to get such estimates is to assume that H is *Lipschitz continuous,* which means that there exists a constant L (called the Lipschitz constant) so that

$$|H(c,d) - H(a,b)| \le L\sqrt{(c-a)^2 + (d-b)^2} \qquad \text{for all } a,\ b,\ c,\ d.$$

This is case if H has partial derivatives satisfying $|\partial H/\partial t| \le L$ and $|\partial H/\partial s| \le L$, so being Lipschitz continuous is not at all uncommon.

For fixed s, we estimate the length of the path $t \mapsto H(t,s)$ by noting that

$$\sum_{j=1}^{n} |H(t_j,s) - H(t_{j-1},s)| \le \sum_{j=1}^{n} L\cdot(t_j - t_{j-1}) = L\cdot(t_n - t_0),$$

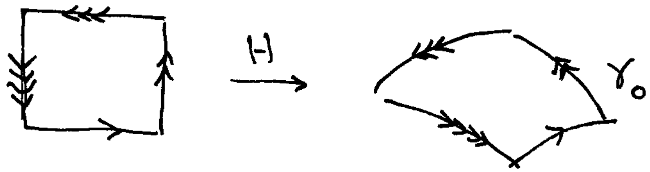
so the path is at most L times the length of the parameter interval. The same argument holds for paths $s \mapsto H(t,s)$, and by combining these result we get the same result for paths created as the boundary of subrectangles such as above.

We shall call a function f *analytic* in a region Ω if its derivative f' exists at all points in Ω. Note that we do *not* require f' to be continuous. That will turn out to be a consequence of analyticity.

22 Theorem. (Cauchy–Goursat) *Assume f is analytic in a region Ω, and let $H\colon [0,1]\times[0,1] \to \Omega$ be a homotopy. Let γ_0 be the curve obtained from H by traversing the boundary of the square $[0,1]\times[0,1]$ once in the positive direction, as explained above. If γ_0 is a path (i.e., has finite length) then*

$$\int_{\gamma_0} f(z)\,dz = 0.$$

Cauchy and Goursat did not prove the theorem in this form, but their versions of it follows easily from this one, and the main idea of the proof below is due to Goursat.

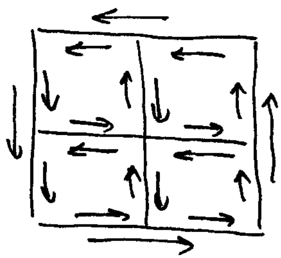


Proof: We prove the theorem first assuming that H is Lipschitz.

Assuming that the integral is not zero, we shall arrive at a contradiction. If the integral is not zero, we can divide f by the value of the integral, so we might as well assume that

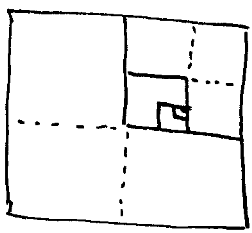
$$\int_{\gamma_0} f(z)\,dz = 1.$$

Now divide $\square_0 = [0,1]\times[0,1]$ into four equal squares. The integral around the



main square is equal to the sum of the four integrals around the four subsquares (the interior parts cancel), so at least one of the four integrals must have absolute value $\geq \frac{1}{4}$. Let $\square_1$ be one such square, and γ_1 the path obtained from H by following the boundary of $\square_1$. So $|\int_{\gamma_1} f(z)\,dz| \geq \frac{1}{4}$.

Next, divide $\square_1$ into four pieces, and apply the same reasoning. We find one of these, call it $\square_2$, so that the corresponding path γ_2 satisfies $|\int_{\gamma_2} f(z)\,dz| \geq \frac{1}{16}$.



Now repeat this: We find squares $\square_0 \supset \square_1 \supset \square_2 \supset \cdots$ each with a path γ_k corresponding to the boundary of $\square_k$, so that

$$\Bigl|\int_{\gamma_k} f(z)\,dz\Bigr| \ge 4^{-k} = 2^{-2k}. \tag{2.3}$$

We shall now turn around and get an upper estimate on the same integral, which contradicts the above.

First, we estimate the length of γ_k. The boundary of $\square_k$ has length $4\cdot 2^{-k}$, and so $\ell(\gamma_k) \le 4L2^{-k}$.

Since all the squares $\square_k$ are compact, there is a point $(t_0, s_0))$ common to them all. And since f is differentiable at $z_0 = H(t_0, s_0)$, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + e(z)(z - z_0), \qquad \lim_{z\to z_0} e(z) = 0.$$

But $\int_{\gamma_k}\bigl(f(z_0) + f'(z_0)(z - z_0)\bigr)\,dz = 0$, so that $\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} e(z)(z - z_0)\,dz$. Now $|z - z_0| \le 2^{-k}\sqrt{2}L$ for z on γ_k, and given $\varepsilon > 0$, we get $|e(z)| < \varepsilon$ along γ_k if k is large enough. Then

$$\begin{aligned}\Bigl|\int_{\gamma_k} f(z)\,dz\Bigr| &= \Bigl|\int_{\gamma_k} \bigl(e(z)(z - z_0)\bigr)\,dz\Bigr| \\ &\le \ell(\gamma_k)\varepsilon 2^{-k}\sqrt{2} \le 4L2^{-k}\varepsilon 2^{-k}\sqrt{2} = 4\sqrt{2}\varepsilon L2^{-2k},\end{aligned}$$

which contradicts (2.3) if $\varepsilon < 1/(4\sqrt{2}L)$. This finishes the proof when H is Lipschitz.

In the more general case when H is *not* Lipschitz, then it can easily be approximated by Lipschitz homotopies: Given any partition $0 = t_0 < t_1 < \cdots < t_n = 1$, put $\tilde{H}(t_j, t_k) = H(t_j, t_k)$ for $j, k = 0, 1, \ldots, n$ and interpolate in each subrectangle $[t_{j-1}, t_j] \times [t_{k-1}, t_k]$:

$$\begin{aligned}&\tilde{H}\bigl(ut_j + (1-u)t_{j-1}, vt_k + (1-v)t_{k-1}\bigr) \\ &\qquad = uvH(t_j, t_k) + u(1-v)H(t_j, t_{k-1}) \\ &\qquad\qquad + (1-u)vH(t_{j-1}, t_k) + (1-u)(1-v)H(t_{j-1}, t_{k-1})\end{aligned}$$

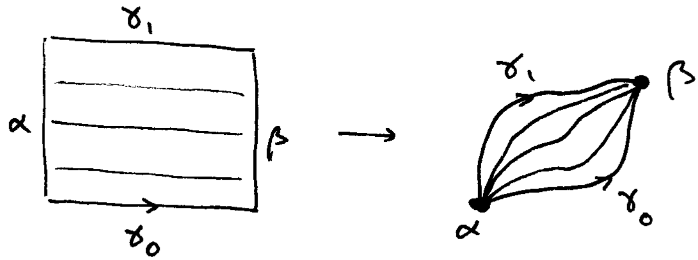
where $u, v \in [0, 1]$. This is Lipschitz, so the conclusion of the theorem is true when H is replaced by $\tilde{H}$. The boundary curves of $\tilde{H}$ yield broken line approximations to the boundary curves of H and the result follows by going to the limit as the partition is refined. ∎

Homotopies with fixed end points. Consider two paths γ_0 and γ_1 in Ω, with the same starting and ending points:[3]

$$\gamma_0(0) = \gamma_1(0) = \alpha, \quad \gamma_0(1) = \gamma_1(1) = \beta.$$

They will be called *homotopic with fixed end points* in Ω if there exists a homotopy H so that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma_1(t), \quad H(0,s) = \alpha, \quad H(1,s) = \beta, \qquad \text{for all } t \text{ and } s.$$



23 Corollary. *If γ_0 and γ_1 are homotopic with fixed end points in Ω and f is analytic in Ω then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof: The integral around the curve corresponding to the boundary of $[0,1] \times [0,1]$ consists of four parts: γ_0 in the forward direction, γ_1 in the reverse direction, and two parts (corresponding to $t = 0$ and $t = 1$) which are degenerate curves staying still at α and β. So the integrals along the latter two are zero, and what remains is $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$. ∎

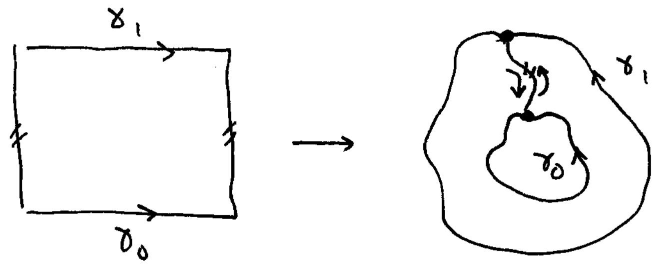
Homotopies via closed paths. Now consider two closed paths γ_0 and γ_1 in Ω:

$$\gamma_0(0) = \gamma_0(1), \quad \gamma_1(0) = \gamma_1(1).$$

They will be called *homotopic via closed paths* in Ω if there exists a homotopy H so that

$$H(0,t) = \gamma_0(t), \quad H(1,t) = \gamma_1(t), \quad H(s,0) = H(s,1), \qquad \text{for all } s \text{ and } t.$$

[3] There is no loss of generality to assume that all paths are parametrized on the interval $[0,1]$, and it simplifies the notation in many proofs.



24 Corollary. *If γ_0 and γ_1 are homotopic via closed paths in Ω and f is analytic in Ω then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof: The proof is just like the corresponding proof for fixed end points. The difference is that now, the two paths corresponding to $t = 0$ and $t = 1$ are not stationary, but one is the reverse of the other, so their integrals cancel. (Actually, there is a small problem here, in that the latter two paths may not have finite lengths, i.e., they may not be paths at all, merely curves. But in the approximation argument used in the proof of the Cauchy–Goursat theorem, the approximations to these two curves will be opposites, so their integrals cancel. And then it does not matter that those integrals may not converge as the mesh size goes to zero.) ∎

A region Ω is called *simply connected* if any two closed paths in Ω are homotopic in Ω via closed paths. Equivalently, any two paths with the same starting and ending points are homotopic with fixed endpoints. Then the integral of an analytic function on Ω is independent of the path, so we can write $\int_\gamma f(z)\,dz = \int_\alpha^\beta f(z)\,dz$ where γ starts in α and ends in β. In particular, f has an antiderivative F which can be defined by $F(z) = \int_\alpha^z f(\zeta)\,d\zeta$ for some fixed $\alpha \in \Omega$. Also, $\int_\gamma f(z)\,dz = 0$ for any closed path γ in Ω.

The logarithm, revisited. Let Ω be any simply connected region *not* containing 0. Then the function $z \mapsto 1/z$ has an antiderivative in Ω. If $1 \in \Omega$, it we may use it as a starting point and define

$$F(z) = \int_1^z \frac{d\zeta}{\zeta}, \quad \text{so that } F'(z) = \frac{1}{z}.$$

From this we find using the chain rule:

$$\frac{d}{dz}\bigl(ze^{-F(z)}\bigr) = e^{-F(z)} - z\frac{e^{-F(z)}}{z} = 0,$$

so that $ze^{-F(z)}$ is a constant. Evaluating at $z=1$ we find that this constant is 1, and so $e^{F(z)}=z$ for all z. In other words, F is a branch of the logarithm, and we can write

$$\ln z = \int_1^z \frac{d\zeta}{\zeta}.$$

If Ω does not contain 1, we just pick any starting point $\alpha\in\Omega$ and any β so that $e^\beta=\alpha$, and we can then put

$$\ln z = \beta + \int_\alpha^z \frac{d\zeta}{\zeta}.$$

Again, this will define a branch of the logarithm in Ω, and it is completely determined by the requirement that $\ln\alpha=\beta$.

The most common way to create a simply connected region on which to define a branch the logarithm is to introduce a *branch cut*, which is just a simple curve starting at 0 and going off to infinity. (This curve must be parametrized on a half open interval in order to be able to continue to infinity, so it's a little different from curves previously considered. Most commonly a half line is used, but any curve will do, and sometimes unorthodox choices are useful.) One then lets Ω be all of $\mathbb{C}$ with the points on the branch cut removed.

Sometimes the branch defined by $\ln z = \int_1^z \frac{d\zeta}{\zeta}$ will be called the *principal* branch of the logarithm in Ω, although this term is most commonly used when the branch cut is the negative real axis.

Winding number. Consider now a path $\gamma\colon [a,b]\to\mathbb{C}\setminus\{z_0\}$, with $\gamma(a)=\alpha$, $\gamma(b)=\beta$. Assuming it lies in some simply connected subregion of $\mathbb{C}\setminus\{z_0\}$, we can write

$$\int_\gamma \frac{dz}{z-z_0} = \ln(\beta-z_0)-\ln(\alpha-z_0).$$

Taking the exponential of this, we find

$$\exp\int_\gamma \frac{dz}{z-z_0} = \frac{\beta-z_0}{\alpha-z_0}, \tag{2.4}$$

a result which is *independent* of the choice of branch for the logarithm.

In general, a path in $\mathbb{C}\setminus\{z_0\}$ can be divided into a finite number of subpaths for which the above reasoning holds, and so we can multiply together the results (2.4) for the individual subpaths and get the same formula (2.4) for the whole path. In particular, if γ is closed then $\exp\int_\gamma 1/(z-z_0)\,dz=1$, so the integral is an integer multiple of $2\pi i$. In other words, we have proved

25 Proposition. *If γ is a closed path in $\mathbb{C}\setminus\{z_0\}$ then the number $\operatorname{ind}_\gamma(z_0)$ defined by*

$$\operatorname{ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}$$

is an integer.

$\operatorname{ind}_\gamma(z_0)$ is called the *index* of z_0 with respect to γ, or the *winding number* of γ around z_0. It quite literally measures the number of times γ winds around z_0 in the positive direction. From the integral theorem we know that homotopic paths have the same winding number, where of course the homotopies must be via closed paths in $\mathbb{C}\setminus\{z_0\}$. This allows us to prove a classical result:

26 Theorem. (The fundamental theorem of algebra) *Any nonconstant complex polynomial has at least one complex root.*

Proof: Assume that $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has no complex root, where $n \ge 1$. Then the paths given by $\gamma_r(t) = p(re^{it})$ where $0 \le t \le 2\pi i$ are all homotopic in $\mathbb{C}\setminus\{0\}$, and since γ_0 is a constant path, $\operatorname{ind}_{\gamma_r}(0) = \operatorname{ind}_{\gamma_0}(0) = 0$ for all $r > 0$.

Now write $\sigma_s(t) = s^n p(s^{-1}e^{it})$ for $s > 0$. Clearly σ_s is just a rescaled version of $\gamma_{1/s}$, so $\operatorname{ind}_{\sigma_s}(0) = \operatorname{ind}_{\gamma_{1/s}}(0) = 0$. But

$$\sigma_s(t) = e^{int} + sa_{n-1}e^{i(n-1)t} + \cdots + s^n a_0,$$

which also makes sense for $s = 0$, and all the paths σ_s are homotopic. Thus $\operatorname{ind}_{\sigma_0}(0) = 0$. But $\sigma_0(t) = e^{int}$, and a direct calculation shows that $\operatorname{ind}_{\sigma_0}(0) = n$. This is a contradiction, and the proof is complete. ∎

Cauchy's integral formula

Let Ω be a region, and $z \in \Omega$. If the disk $B_\rho(z)$ is contained in Ω, then all circles centered at z and with radius $< r$ are homotopic via closed paths in $\Omega\setminus\{z\}$: More precisely, let the circle γ_r be given by

$$\gamma_r(t) = z + re^{it}, \qquad t \in [0, 2\pi]$$

and then that same formula provides the homotopy:[4]

$$H(t,r) = z + re^{it}, \qquad r \in [r_1, r_2],\ t \in [0, 2\pi]$$

defines a homotopy between γ_{r_1} and γ_{r_2}.

[4] It was easier, when we proved general theorems on homotopies, to assume they were defined on $[0,1]\times[0,1]$. But nothing changes if we allow them to be defined on more general rectangles.

27 Theorem. (Cauchy's integral formula) *Assume that γ is a closed curve in Ω which is homotopic via closed paths in $\Omega\setminus\{z\}$ to one (and hence all) of the above small circles. If f is analytic in Ω, then*

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Proof: The integrand is an analytic function of ζ in $\Omega\setminus\{z\}$. Therefore the integral is unchanged if we replace γ by an arbitrarily small circle γ_r around z.

Direct calculation shows that

$$\int_\gamma \frac{1}{\zeta-z}\,d\zeta=2\pi i.$$

Multiplying this by $f(z)$ (which is independent of ζ, and therefore can be put inside the integral), we see that we only need to prove

$$\lim_{r\to 0}\int_{\gamma_r}\frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta=0.$$

(The limit seems unnecessary, but is harmless, since the integral does not depend on r when r is small.)

But f is continuous, so $|f(\zeta)-f(z)|<\varepsilon$ for ζ on γ_r if r is small enough. Then the whole integrand has absolute value smaller than ε/r, and the integration path has length $2\pi r$, so the integral is smaller than $\varepsilon/r\cdot 2\pi r=2\varepsilon\pi$. This completes the proof. ∎

From this we can already deduce the following. Recall that a function is called *entire* if it analytic in all of the complex plane.

28 Theorem. (Liouville) *A bounded, entire function is constant.*

Proof: Assume that $|f(z)|\le M$ for all $z\in\mathbb{C}$. Let $z\in\mathbb{C}$. If $r>|z|$ then

$$\begin{aligned}f(z)-f(0)&=\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)}{\zeta}\,d\zeta\\&=\frac{1}{2\pi i}\int_{\gamma_r}f(\zeta)\Big(\frac{1}{\zeta-z}-\frac{1}{\zeta}\Big)\,d\zeta=\frac{1}{2\pi i}\int_{\gamma_r}f(\zeta)\frac{z}{(\zeta-z)\zeta}\,d\zeta\end{aligned}$$

so that

$$|f(z)-f(0)|\le\frac{2\pi r}{2\pi}M\frac{|z|}{(r-|z|)r}\to 0\qquad\text{as } r\to 0.$$

This shows that $f(z)=f(0)$, and completes the proof. ∎

29 Theorem. (Cauchy's generalized integral formula) *Under the assumptions of theorem 27, f is infinitely differentiable, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \qquad n = 0, 1, 2, \ldots \tag{CF_n}$$

Proof: For $n = 0$, this is just the standard Cauchy formula. (I shall refer to it as (CF_0) from now on.)

For $n = 1$, first notice that

$$\frac{1}{\zeta - w} - \frac{1}{\zeta - z} = \frac{w - z}{(\zeta - w)(\zeta - z)} \tag{2.5}$$

so that two applications of (CF_0) followed by (2.5) yields

$$\frac{f(w) - f(z)}{w - z} = \frac{1}{2\pi i(w - z)}\int_\gamma \Bigl(\frac{1}{\zeta - w} - \frac{1}{\zeta - z}\Bigr) f(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - w)(\zeta - z)}\,d\zeta.$$

Now let $w \to z$. It is at least plausible that the righthand side above should converge to the righthand side of (CF_1), since the integrand converges. However, pointwise convergence of the integrands is not quite enough. We need to estimate the difference. So from we just proved, together with another applicaton of (2.5) we find

$$\frac{f(w) - f(z)}{w - z} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta = \frac{w - z}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - w)(\zeta - z)^2}\,d\zeta.$$

And here the integral on the right remains bounded so long as w and z remain in a bounded region and stay away from γ, so the whole expression goes to zero as $w \to z$. This proves (CF_1).

It will be important in a short while that we have proved (CF_1) only assuming the regular Cauchy formula (CF_0): Other than that, we have in fact not used the assumption that f is analytic. So we have shown that Cauchy's integral formula *implies* that f is analytic.

We shall show that f' is continuous next. To this end, we note that two applications of (CF_1) yield

$$\begin{aligned} f'(w) - f'(z) &= \frac{1}{2\pi i}\int_\gamma f(\zeta)\Bigl(\frac{1}{(\zeta - w)^2} - \frac{1}{(\zeta - z)^2}\Bigr)\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{(2\zeta - w - z)}{(\zeta - w)^2(\zeta - z)^2}\,d\zeta \cdot (w - z), \end{aligned}$$

and again, the integral is bounded so long as w and z stay in a bounded region and away from γ, and the continuity of f' follows from that.

Next we note that, for any natural number $n \ge 0$,

$$\frac{d}{d\zeta}\left(\frac{f(\zeta)}{(\zeta-z)^n}\right) = \frac{f'(\zeta)}{(\zeta-z)^n} - n\frac{f(\zeta)}{(\zeta-z)^{n+1}}.$$

We now integrate this around γ, applying Proposition 21 with $F(\zeta) = f(\zeta)/(\zeta-z)^n$ to obtain the useful formula[5]

$$n\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta = \int_\gamma \frac{f'(\zeta)}{(\zeta-z)^n}\,d\zeta. \tag{2.6}$$

We first substitute it, with $n=1$, into (CF_1) to get

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{\zeta-z}\,d\zeta,$$

that is, Cauchy's integral formula for f'. But as we noted above, that implies that f' itself is analytic.

So far we have proved that if f is analytic then so is f'. It follows by induction that f'' exists and is analytic, and so on by induction, so f is indeed infinitely differentiable.

It only remains to prove (CF_n) by induction on n. We already know that (CF_0) and (CF_1) are true. So assume that (CF_{n-1}) holds. Apply it to f' rather than to f itself (clearly, the $(n-1)^{\text{st}}$ derivative of f' is $f^{(n)}$) and then use (2.6):

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f'(\zeta)}{(\zeta-z)^n}\,d\zeta = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta,$$

which is (CF_n). The proof is complete. ∎

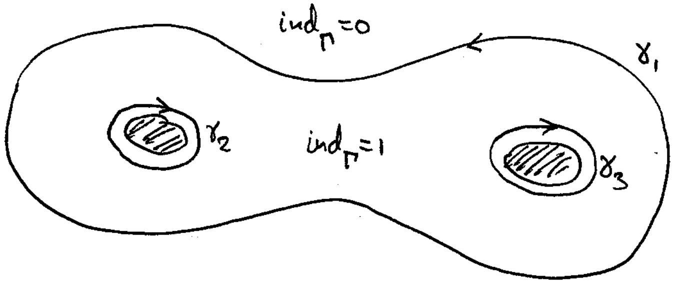
The generalized Cauchy formula is quite easy to remember, should you ever forget it: It is the result of differentiating the ordinary Cauchy formula (CF_0) n times with respect to z, assuming that we can move the differentiation operator inside the integral sign. This is permitted under fairly general conditions, which are satisfied here, but we preferred a more direct approach. Our proof of (CF_1) from (CF_0) is in fact a special case of this more general result on differentiation inside the integral, done by explicit calculation rather than general theory. That we could proceed to higher n by partial differentiation seems somewhat like a fortuitous accident, but one that we were happy to exploit.

[5] What we just did is nothing other than partial integration.

The global Cauchy integral formula

We sometimes need to work with the sum of integrals around several closed paths. A bit of notation is helpful. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a finite collection of closed paths, we may call Γ a closed *multipath*.

Write $\int_\Gamma f(z)\,dz = \sum_{k=1}^n \int_{\gamma_k} f(z)\,dz$ and $\operatorname{ind}_\Gamma(z) = \sum_{k=1}^n \operatorname{ind}_{\gamma_k}(z)$. Obviously, we will say a point is on Γ if it is on one of the paths γ_k.



30 Theorem. (Cauchy's formula, global (holonomy) version) *Assume that Ω is a region, and that Γ is a closed multipath in Ω so that* $\operatorname{ind}_\Gamma(z) = 0$ *for any $z \notin \Omega$. If f is analytic in Ω then for any $z \in \Omega$ that is not on Γ,*

$$\operatorname{ind}_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof: [This proof is due to J. D. Dixon (1971).] Inserting the definition of the index on the left hand side, we see that what we must prove is

$$\int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = 0.$$

The integrand here is more regular than it looks: We can define a continuous function g on $\Omega \times \Omega$ by

$$g(z,\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \neq z, \\ f'(z), & \zeta = z, \end{cases}$$

In fact, when w and z are close together we can surround them both by a closed path γ and write

$$g(z,\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-\zeta)(w-z)}\,dw,$$

as noted in the proof of the generalized Cauchy formula. The continuity of g is easily deduced from this formula. Now define the function h on $\mathbb{C}$ by

$$h(z) = \begin{cases} \displaystyle\int_\Gamma g(z,\zeta)\,d\zeta, & z \in \Omega, \\ \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, & z \in \mathbb{C} \text{ not on } \Gamma \text{ and } \operatorname{ind}_\Gamma(z) = 0. \end{cases}$$

Note that for some z both cases in the above definition apply, but for those z the two definitions agree because of the assumption $\operatorname{ind}_\Gamma(z) = 0$ and the definition of the index.

The assumption that the index is zero for points outside Ω implies that at least one of the cases apply for any z, so h is indeed defined in the entire plane. Moreover each of the two parts of the definition defines an analytic[6] function in an open set, so h is entire.

Finally, when $|z|$ is large then $\operatorname{ind}_\Gamma(z) = 0$ so the second part of the definition applies, and a simple estimate shows that $h(z) \to 0$ as $z \to \infty$. In particular h is bounded, so it is constant by Liouville's theorem, and clearly this constant must be zero, so $h(z) = 0$ for all z. This completes the proof. ∎

Earlier on, we proved Cauchy's integral formula from the integral theorem. We now get to do it in reverse:

31 Corollary. (Cauchy's integral theorem, global (holonomy) version) *Assume that Ω is a region, and that Γ is a closed multipath in Ω so that* $\operatorname{ind}_\Gamma(z) = 0$ *for any $z \notin \Omega$. If f is analytic in Ω*

$$\int_\Gamma f(\zeta)\,d\zeta = 0.$$

Proof: Pick any point $z \in \Omega$ that is not on Γ, define $g(\zeta) = (\zeta - z) f(\zeta)$, and apply the global Cauchy formula (CF_0) to g. ∎

That the generalized Cauchy formula also generalizes to the global setting in the form

$$\operatorname{ind}_\Gamma(z) f^{(n)}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

should not come as a big surprise. It can be proved just like Theorem 29. The details are left to the reader.

[6] This claim requires a bit more work. Perhaps better left until later.

Chapter 3

The classification of isolated singularities

It seems to me that the book's treatment of isolated singularities is organized in a somewhat confusing fashion. I'll try to simplify.

Let f be a function which is analytic in a neighbourhood of some point z_0, *except* at the point z_0 itself. Then z_0 is called an *isolated singularity* of f.

Recall that under the stated assumption, f can be represented by its *Laurent series* at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad 0 < |z-z_0| < R$$

where the outer radius of convergence is a positive number (possibly infinite). This immediately leads to the following classification.

Removable singularity. z_0 is called a *removable singularity* of f if $a_n = 0$ for all $n < 0$.

In this case the series above is an ordinary power series, and if we were to (re)*define* $f(z_0) = a_0$ then the redefined function is in fact analytic at z_0. So the "singularity" has disappeared, which is why we called it removable.

To carry this a bit further (we shall need it later), let N be the smallest index n for which $a_n \neq 0$. (If there is none, then f is of course identically zero in a neighbourhood of z_0.) Then we can write

$$\begin{aligned} f(z) &= \sum_{n=N}^{\infty} a_n (z-z_0)^n = (z-z_0)^N \sum_{n=N}^{\infty} a_n (z-z_0)^{n-N} \\ &= (z-z_0)^N \underbrace{\sum_{n=0}^{\infty} a_{n+N} (z-z_0)^n}_{g(z)} = (z-z_0)^N g(z) \end{aligned}$$

where g is analytic at z_0 and $g(z_0) = a_N \neq 0$. If $N > 0$ we call z_0 a *zero of order N* of f.

It is quite clear that, in general, whenever we can write $f(z) = (z-z_0)^N g(z)$ with g analytic and $g(z_0) \neq 0$ and $N > 0$, that z_0 is a zero of order N. (Multiply the Taylor series of g at z_0 by $(z-z_0)^N$ to see this.)

Pole. If there is some $N > 0$ with $a_{-N} \neq 0$ while $a_n = 0$ for all $n < -N$ then we say z_0 is a *pole of order N* of f.

In this case we can write

$$\begin{aligned} f(z) &= \sum_{n=-N}^{\infty} a_n (z-z_0)^n = (z-z_0)^{-N} \sum_{n=N}^{\infty} a_n (z-z_0)^{n+N} \\ &= (z-z_0)^{-N} \underbrace{\sum_{n=0}^{\infty} a_{n-N}(z-z_0)^n}_{g(z)} = \frac{g(z)}{(z-z_0)^N} \end{aligned}$$

where again, g is analytic at z_0 and $g(z_0) = a_{-N} \neq 0$.

The situation is similar to that of a zero of order N: If we can write $f(z) = g(z)/(z-z_0)^N$ with g analytic and $g(z_0) \neq 0$ and $N > 0$, that z_0 is a pole of order N.

Essential singularity. If $a_n \neq 0$ for infinitely many $n < 0$, then z_0 is called an *essential singularity* of f.

How to recognize the three kinds of singularity, and a bit about their properties. You don't actually need the Laurent series to recognize the different kinds of singularity.

32 Proposition. *z_0 is a removable singularity if and only if $|f|$ is bounded in some neighbourhood of z_0.*

Proof: The "only if" part is quite obvious: If z_0 is a removable singularity then f is in fact analytic at z_0 (after a suitable redefinition at the single point z_0), and analytic functions, being continuous, are locally bounded.

On the other hand, if $|f|$ is bounded near z_0, define $g(z) = (z-z_0)^2 f(z)$ for $z \neq z_0$ and $g(z_0) = 0$. Then g is analytic at $z \neq z_0$, but also $g'(z_0) = 0$ by direct definition of the derivative. So g is in fact analytic at z_0, and we can write

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

for z in a neighbourhood of z_0. Now $b_0 = g(0) = 0$ and $b_1 = g'(0) = 0$, so

$$g(z) = (z-z_0)^2 \sum_{n=2}^{\infty} b_n (z-z_0)^{n-2} = (z-z_0)^2 \underbrace{\sum_{n=0}^{\infty} b_{n+2}(z-z_0)^n}_{f(z)},$$

i.e., the indicated sum must be $f(z)$, which therefore has a removable singularity at z_0. ∎

33 Proposition. *f has a pole at z_0 if and only if $f(z) \to \infty$ as $z \to z_0$.*

Proof: Recall that $f(z) \to \infty$ really means $|f(z)| \to \infty$.

Again, the "only if" part is obvious, for if z_0 is a pole then we can write $f(z) = g(z)/(z-z_0)^N$ where g is analytic with $g(z_0) \neq 0$ and $N > 0$, so $f(z) \to \infty$ follows.

On the other hand, assume that $f(z) \to \infty$ as $z \to z_0$. Now define $h(z) = 1/f(z)$ for $z \neq z_0$. From the assumption it follows that h is bounded in a neighbourhood of z_0 so it has a removable singularity at z_0. Therefore we can write $h(z) = (z-z_0)^N g(z)$ with g analytic, and $g(z_0) \neq 0$, and $N \geq 0$. So $f(z) = 1/\big((z-z_0)^N g(z)\big)$. Since $f(z) \to \infty$ we must in fact have $N > 0$, and so z_0 is a pole. ∎

If course, it follows from the three alternatives and the above two characterizations that z_0 is an essential singularity if and only if $f(z)$ is neither bounded nor goes to infinity as $z \to z_0$. This indicates some rather "wild" behaviour of the function. In fact, more is true:

34 Proposition. *If z_0 is an essential singularity of f then, for every $\alpha \in \mathbb{C}$ and every neighbourhood of z_0, we can find z in that neighbourhood so that $f(z)$ is arbitrarily close to α.*

Proof: We prove the contrapositive. Assume there is some $\alpha \in \mathbb{C}$ and a neighbourhood of z_0 so that $f(z)$ can not get arbitrarily close to α for z in that neighbourhood. But then the function $h(z) = 1/\big(f(z)-\alpha\big)$ is bounded in the given neighbourhood, and therefore it has a removable singularity at z_0. Define $h(z_0)$ so that h is analytic at z_0. Then

$$f(z) = \alpha + \frac{1}{h(z)},$$

so if $h(z_0) \neq 0$ then f has a removable singularity at z_0 by Proposition 32, and if $h(z_0) = 0$ then f has a pole at z_0 by Proposition 33. In neither case does f have an essential singularity at z_0. ∎

A unified theory of zeros and poles. We could say that f has *order n* at z_0 if we can can write f on one of these two equivalent forms in some punctured neighbourhood[1] of z_0:

$$f(z) = \sum_{k=n}^{\infty} a_k (z-z_0)^k = (z-z_0)^n g(z), \qquad a_n = g(z_0) \neq 0$$

[1] A *punctured neighbourhood* of z_0 is a neighbourhood where z_0 itself has been excluded.

where g is analytic at z_0. Thus, if $n > 0$ this means that f has a zero of order n at z_0, while if $n < 0$ it means that f has a pole of order $-n$ at z_0. Finally, if $n = 0$ it simply means that f is analytic at z_0 with $f(z_0) \neq 0$ – which is surely going to be the commonest case by far, unless f is identically zero (more on that below).

If we write[2] $\operatorname{ord}(f, z_0)$ for the order of f at z_0, we can easily deduce formulas like $\operatorname{ord}(fg, z_0) = \operatorname{ord}(f, z_0) + \operatorname{ord}(g, z_0)$ and $\operatorname{ord}(f/g, z_0) = \operatorname{ord}(f, z_0) - \operatorname{ord}(g, z_0)$. These provide handy rules for dealing with products and quotients where the factors have zeros and poles of varying order. For example, $\sin z/(1-\cos z)$ has a single pole at the origin, because $\sin z$ has a single zero, $1-\cos z$ a double zero, and $1-2=-1$.

On isolated zeros. Above we found that if $f(z_0) = 0$, then can write $f(z) = (z-z_0)^N g(z)$ where g is analytic, $g(z_0) \neq 0$ and $N > 0$, or else $f(z)$ is identically zero in some neighbourhood of z_0. In the former case, we call z_0 an *isolated zero* of f.

In other words, if f is analytic at z_0 then either $f(z)$ is identically zero in some neighbourhood of z_0 or else $f(z) \neq 0$ for all z in some neighbourhood of z_0, with the possible exception of $z = z_0$ itself.

Now let f be analytic in a region Ω. (And recall that a region is, by definition, open and connected.) If we write A for the set of points in Ω that are either isolated zeros, or not zeros at all, and B for those points where f is identically zero in some neighbourhood, then A and B are both open subsets of Ω. They are also disjoint, and their union is all of Ω. Therefore, since Ω is a region, one of the two sets is empty. It follows that *unless f is identically zero in Ω, then all zeros of f in Ω are isolated.*

[2] Beware that this terminology and notation are not standard.

Chapter 4

The Riemann–Lebesgue lemma

This chapter is devoted to a proof of the Riemann–Lebesgue lemma (see p. 504 in the book). This proof is simpler, and the statement stronger, than in the book.

We actually need this greater generality in order to use it in Chernoff's proof of the Fourier representation theorem (see his *Monthly* paper, linked from the course home page).[1]

We use the following notation for the nth Fourier coefficient of a 2π-periodic function f:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

35 Lemma. (Riemann–Lebesque) *Assume that f is 2π-periodic, bounded and integrable. Then $\hat{f}(n) \to 0$ when $n \to \pm\infty$.*

Proof: We shall prove this only for real-valued functions. If f is complex-valued, the result will follow from the result applied to the real and imaginary parts of f separately.

First, we prove the result for an extremely special case: Namely, a *single step*, which is a function of the form

$$s(x) = \begin{cases} 1 & a+2k\pi \le x \le b+2k\pi, \quad k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

where $a < b$ and $b - a < 2\pi$. Then

$$\hat{s}(n) = \frac{1}{2\pi}\int_a^b e^{-inx}\,dx = \frac{e^{-inb} - e^{-ina}}{2\pi i n} \to 0 \qquad \text{as } n \to \pm\infty$$

since the numerator is bounded and the denominator goes to infinity.

Second, since any step function is a linear combination of a finite number of single steps, the same result holds for step functions.

[1]There exists an even more general statement that is beyond us at this point: It requires the use of the Lebesgue integral, which is more general than the Riemann integral that is introduced in calculus.

Finally, now assume that f is integrable, and pick any $\varepsilon > 0$. It follows – practically direct from the definition of integrability – that there exists a step function s with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s(x)|\,dx < \varepsilon.$$

From this we get

$$|\hat{f}(n) - \hat{s}(n)| = \frac{1}{2\pi}\left|\int_{-2\pi}^{\pi}\bigl(f(x) - s(x)\bigr)e^{-inx}\,dx\right| \le \frac{1}{2\pi}\int_{-2\pi}^{\pi}|f(x) - s(x)|\,dx < \varepsilon$$

as well. We have shown that $\hat{s}(n) \to 0$, so there is some N so that $|n| \ge N$ implies $|\hat{s}(n)| < \varepsilon$. Whenever $|n| \ge N$, then

$$|\hat{f}(n)| \le |\hat{f}(n) - \hat{s}(n)| + |\hat{s}(n)| < \varepsilon + \varepsilon = 2\varepsilon,$$

which finishes the proof. ∎

Notice that the Riemann–Lebesgue lemma says nothing about *how fast* $\hat{f}(n)$ goes to zero. With just a bit more of a regularity assumption on f, we can show that $\hat{f}(n)$ behaves roughly like $1/n$ or better. This is easy if f is continuous and piecewise smooth, as is seen from the identity $\widehat{f'}(n) = in\hat{f}(n)$, which arises from partial integration. Applying the Riemann–Lebesgue lemma to f' we conclude that $\hat{f}(n)$ is $1/n$ times something that goes to zero, so $\hat{f}(n) \to 0$ *faster* than $1/n$.

We can even drop the requirement of continuity: Just so long as f is piecewise smooth, partial integration yields a formula just like $\widehat{f'}(n) = in\hat{f}(n)$, with the addition of some extra terms coming from the points of discontinuity. But these extra terms are bounded, so this time we get $\hat{f}(n) \to 0$ *as fast* as $1/n$.

If f has more continuous derivatives, we can keep on differentiating: We get $\widehat{f^{(k)}}(n) = (in)^k\hat{f}(n)$, and conclude that $\hat{f}(n)$ goes to zero faster than n^{-k}.