75042 Partial differential equations

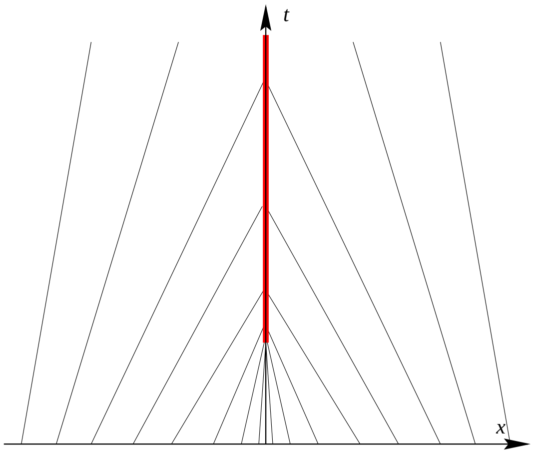
Problem set #12 — suggested solutions

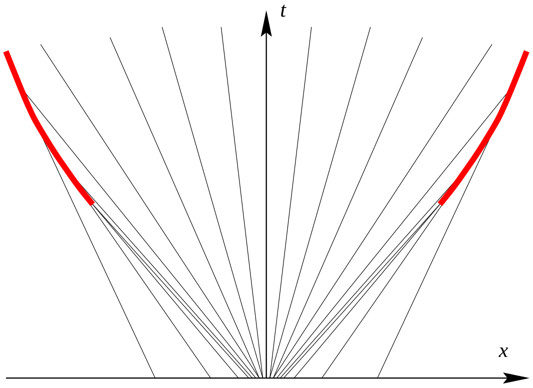
#11.2: (Continued from last week. There was a mistake in last week's solution: In case (ii) the shocks are at $x = \pm 8\pi^3/27$, not at $\pm 8\pi/27$.)

In cases (iii) and (iv) write the initial data as $u_0(\xi) = \pm 2 \operatorname{arctg} \xi$. The characteristic starting at $(\xi, 0)$ has speed $f'(\xi) = \pm c(\xi)$ where c is the function shown in the problem statement. That characteristic has the equation $x = \xi \pm c(\xi)t$. This yields $x_\xi = 1 \pm c'(\xi)t$, and a shock appears when this quantity first becomes zero. That is, it happens when $\pm c'(\xi)$ is negative and as small as possible. The corresponding time is $t = \mp 1/c'(\xi)$.

In case (iii) we use the $+$ sign so the minimum is at $\xi = 0$, and the shock appears for $\xi = 0$ at $t = -1/c'(0) = 1/(2\pi^2)$ ($t \approx 1/19.7$ from the picture). All characteristics from both sides will hit this shock, which gets speed 0 (use symmetry) and hence has the equation $x = 0$ for $t > 1/(2\pi^2)$.

In case (iv) we use the $-$ sign and the minimum of $-c'$ is at $\xi \approx \pm 2$. The shocks appear at $t = 1/c'(\xi) \approx 1/2$ and $x = \xi - c(\xi)t \approx \pm(2 + 10/2) = \pm 7$. For large t we expect the shocks to move with speeds approaching $\pm 8\pi^3/27$ as in case (ii). The characteristics which started near $\xi = 0$ with speeds in the interval $[-8\pi^3/27, 8\pi^3/27]$ will never catch up with a shock, and this part of the solution is like the expansion wave from case (ii). All the other characteristics will sooner or later hit one of the two shocks.



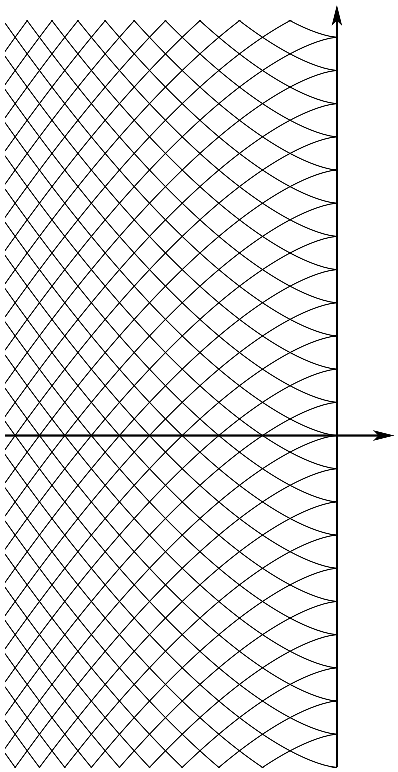


The pictures show shocks and characteristic in case (iii) (left picture) and (iv) (right picture). The two axes are not equally scaled, and the pictures are far from exact — they are not computed, but "handmade" using a drawing program.

#12.1: The Tricomi equation $u_{xx} + xu_{yy} = 0$ has characteristic condition $\xi^2 + x\eta^2 = 0$, with no nonzero solutions if $x > 0$, a unique solution up to scaling, given by $\xi = 0$, if $x = 0$, and two solutions, up to scaling, given by $\xi = \pm\sqrt{-x}\,\eta$ if $x < 0$.

Thus the equation is elliptic if $x > 0$, parabolic if $x = 0$, and hyperbolic if $x < 0$.

In the hyperbolic area the characteristics are orthogonal to $(\pm\sqrt{-x}, 1)$ (this equation comes from setting $\eta = 1$ above). Thus these characteristics have tangent vectors parallell to $(1, \mp\sqrt{-x})$, so they satisfy $\mathrm{d}y = \mp\sqrt{-x}\,\mathrm{d}x$. Integrate that to get $y - y_0 = \pm\frac{2}{3}(-x)^{3/2}$. These curves — for various values of y_0 — are plotted in the figure on the right.



No more problems. This is it!