75042 Partial differential equations

Problem set #9 — suggested solutions

#8.6: (left over from last week)

(a) We solve $xu_x + yu_y = \lambda u$ with initial data $u(x, 1) = g(x)$. The characteristic equations are

$$\dot{x} = x, \quad \dot{y} = y, \quad \dot{z} = \lambda z$$

with solutions $x = Ae^t$, $y = Be^t$, $z = Ce^{\lambda t}$ (no integration constants thrown away this time). Eliminating t we get (using new constants)

$$x = \alpha y, \quad z = \beta x^\lambda$$

for the characteristics. It should be noted that no characteristic curve actually reaches $x = y = 0$. The characteristic curves starting at points with $y = 1$ can only reach the upper half plane therefore, so we only get a determined solution for $y > 0$.

The initial condition means that our characteristic curve should have $z = g(x)$ when $y = 1$, which happens at $x = \alpha$. So we get $\beta\alpha^\lambda = g(\alpha)$, that is $\beta = \alpha^{-\lambda} g(\alpha)$. Thus we get finally for our solution

$$u(x, y) = z = \alpha^{-\lambda} g(\alpha) x^\lambda = \left(\frac{x}{\lambda}\right)^\lambda g\left(\frac{x}{y}\right) x^\lambda = y^\lambda g\left(\frac{x}{y}\right).$$

(b) We solve $uu_x + u_y = 1$ with data $u(x, x) = x/2$. The characteristic equations are

$$\dot{x} = z, \quad \dot{y} = 1, \quad \dot{z} = 1$$

which we solve for z first, then for x and y, giving $z = t$ (we discarded the integration constant), $y = t + A$, and $x = t^2/2 + B$. Eliminating t we end up with

$$x = \frac{1}{2} z^2 + B, \quad y = z + A \tag{1}$$

for the general characteristic of the given equation.

The initial data says $z = x/2$ when $x = y$. To fit this with (1), we put $x = y = 2z$ to get

$$2z = \frac{1}{2} z^2 + B, \quad 2z = z + A.$$

The second equation implies $z = A$, which we substitute in the first equation to obtain

$$B = 2A - \frac{1}{2} A^2.$$

Finally we substitute this B value back into (1), use the second half of the equation to write $A = y - z$, which is also substituted in the first half, with the result

$$x = \frac{1}{2} z^2 + 2(y - z) - \frac{1}{2}(y - z)^2 = 2y - 2z - \frac{1}{2} y^2 + yz$$

which we solve for z to get the solution

$$u(x, y) = \frac{x - 2y + \frac{1}{2} y^2}{y - 2}.$$

#9.1:

(a) We solve $xu_x + yu_y + u_z = 3u$ with data $u(x, y, 0) = g(x, y)$. The characteristic equations are

$$\dot{x} = x, \quad \dot{y} = y, \quad \dot{z} = 1, \quad \dot{w} = 3w$$

with solutions $x = Ae^t$, $y = Be^t$, $z = t$, and $w = Ce^{3t}$ (where as usual we discarded a constant of integration — for the z variable this time). Eliminating t is trivial:

$$x = Ae^z, \quad y = Be^z, \quad w = Ce^{3z}.$$

Now the initial data says $z = 0$ implies $w = g(x, y)$, and so the previous equations imply $C = g(A, B)$. We end up with

$$u(x, y) = w = g(A, B)e^{3z} = g(xe^{-z}, ye^{-z})e^{3z}.$$

(b) We solve $(y + u)u_x + yu_y = x - y$ with data $u(x, 1) = 1 + x$. The characteristic equations are

$$\dot{x} = y + z, \quad \dot{y} = y, \quad \dot{z} = x - y.$$

The y equation is trivially solved, yielding $y = e^t$ (should have been $y = Ke^t$, but chosing a different K of the same sign just corresponds to a reparametrization, and $K < 0$ would yield a characteristic which never meets the initial curve $y = 1$).

The x and z equations form a linear system which is easily diagonalized. Put $v = (x + z)/2$ and $w = (x - z)/2$ to get $\dot{v} = v$ and $\dot{w} + w = y$. We get $v = Ae^t$ and, since $y = e^t$, $w = e^t/2 + Be^{-t}$. Thus $x = v + w = (A + 1/2)e^t + Be^{-t}$ and $z = v - w = (A - 1/2)e^t - Be^{-t}$. We eliminate t to get

$$x = \Big(A + \frac{1}{2}\Big)y + \frac{B}{y}, \quad z = \Big(A - \frac{1}{2}\Big)y - \frac{B}{y}$$

for the general characteristic. The initial data says that $z = 1 + x$ when $y = 1$. Thus $A - 1/2 - B = 1 + A + 1/2 + B$, so $B = -1$. Inserting this into the equations above, eliminating A and solving for z we get finally

$$u(x, y) = z = x - y + \frac{2}{y}.$$

#9.2: We are asked to show that a function u, defined implicitly by the equation

$$u(x, t) = g\big(x - tF'(u(x, t))\big), \tag{2}$$

satisfies the conservation law $u_t + F(u)_x = 0$.

Differentiating (2) with respect to t we get

$$u_t = g'\big(x - tF'(u)\big)\big(-F'(u) - tF''(u)u_t\big)$$

while applying F to both sides of (2) and differentiating with respect to x yields

$$F(u)_x = F'(u)g'\big(x - tF'(u)\big)\big(1 - tF''(u)u_x\big).$$

Adding (and using $F'(u)u_x = F(u)_x$) we get

$$u_t + F(u)_x = -tF''(u)g'\big(x - tF'(u)\big)\big(u_t + F(u)_x\big)$$

which guarantees that $u_t + F(u)_x = 0$ unless it so happens that $tF''(u)g'\big(x - tF'(u)\big) = -1$. At least for small enough t we should not need to worry about this.

Problem set #10 — for next week

#10.1: Consider "small" waves for the conservation law $u_t + f(u)_x = 0$. By "small" I mean that u should be close to a constant u_0. Discuss the behaviour of the solution. What role is played by the characteristic speed $f'(u)$? (The limit of almost constant u is called the "acoustic limit" in analogy with ordinary soundwaves; a very loud sound at 100 dB means pressure variations on the order of 2 Pa, where atmospheric pressure is about 10^5 Pa.)

#10.2: Discuss the solutions of the (rescaled) traffic problem

$$u_t + \big(u(1-u)\big)_x = 0$$
$$u(x,0) = \begin{cases} u_1 & 0 < x < 1 \\ u_0 & \text{otherwise} \end{cases}$$

In particular, what is the behaviour of the solution as $t \to \infty$?

#10.3: [Exam 1995–08–30]

(a) If you try to solve the initial value problem

$$u_t + \Big(\frac{1}{3}u^3\Big)_x = 0 \qquad (t > 0)$$
$$u(x,0) = x$$

you'll run into difficulty. What is the source of the problem?

(b) Find a weak solution to the initial value problem

$$u_t + \Big(\frac{1}{3}u^3\Big)_x = 0 \qquad (t > 0)$$
$$u(x,0) = \begin{cases} -1 & x \le -1, \\ x & -1 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

You don't need to compute an analytic expression for the shock position(s) after shock(s) have appeared. You don't need to give very stringent proofs for the behaviour of the solutions after this time.