75042 Partial differential equations

Problem set #3 — suggested solutions

#3.1: The given PDE implies $\dot{v} + cv = 0$ where $v(t) = u(x + tb, t)$. Thus $v(t) = Ce^{-ct}$ where the constant C may depend on x. Since the initial data yield $v(0) = g(x)$, we conclude $v(t) = g(x)e^{-ct}$ and therefore $u(x + tb, t) = g(x)e^{-ct}$. Finally replace x by $x - tb$ to obtain the result $u(x, t) = g(x - tb)e^{-ct}$.

#3.2: We find

$$\left| \fint_{\partial B(x,r)} f(\xi)\,\mathrm{d}\xi - f(x) \right| = \left| \fint_{\partial B(x,r)} \big(f(\xi) - f(x)\big)\,\mathrm{d}\xi \right| \le \fint_{\partial B(x,r)} |f(\xi) - f(x)|\,\mathrm{d}\xi < \varepsilon$$

provided $r \le \delta$ and $\delta > 0$ is so that $|f(\xi) - f(x)| < \varepsilon$ whenever $|\xi - x| < \delta$.

#3.3: As in the proof on pp. 29–30, define

$$\varphi(\rho) = \fint_{\partial B(0,\rho)} u(y)\,\mathrm{d}S(y)$$

and find

$$\varphi'(\rho) = \frac{\rho}{n} \fint_{B(0,\rho)} \Delta u(y)\,\mathrm{d}y = -\frac{\rho}{n} \fint_{B(0,\rho)} f(y)\,\mathrm{d}y$$

Now that we know φ' we can integrate to find φ. On one hand (previous problem) $\varphi(0) = u(0)$ (this is what we want to compute), and on the other hand we can find $\varphi(r)$ from the given data. So it's just a question of integrating the derivative:

$$\begin{aligned} u(0) = \varphi(0) &= \varphi(r) - \int_0^r \varphi'(\rho)\,\mathrm{d}\rho \\ &= \fint_{\partial B(0,r)} g(y)\,\mathrm{d}S(y) + \int_0^r \frac{\rho}{n} \frac{1}{\alpha(n)\rho^n} \int_{B(0,\rho)} f(y)\,\mathrm{d}y \end{aligned}$$

Here we must interchange the order of integration: Note that we integrate over those $(y, \rho) \in \mathbb{R}^n \times \mathbb{R}$ satisfying $|y| < \rho < r$.

$$\begin{aligned} u(0) &= \fint_{\partial B(0,r)} g(y)\,\mathrm{d}S(y) + \int_{B(0,r)} f(y) \int_{|y|}^r \frac{1}{n\alpha(n)\rho^{n-1}}\,\mathrm{d}y\,\mathrm{d}\rho \\ &= \fint_{\partial B(0,r)} g(y)\,\mathrm{d}S(y) - \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y) \left(\frac{1}{r^{n-2}} - \frac{1}{|y|^{n-2}} \right) \mathrm{d}y \end{aligned}$$

as desired.

#3.4: Assume u has a local maximum at a point x. Then, applying the usual single variable theory to u as a function of each variable separately, we must conclude $u_{x_i}(x) = 0$ and $u_{x_i x_i}(x) \le 0$. Adding all the latter inequalities together we get $\Delta u(x) \le 0$.

Just in case you thought this solution was too simple, here is a more difficult one, relying on the proof of the mean value property: Define $\varphi(\rho)$ as in the solution to #3.3 above. If $\Delta u(0) > 0$ then $\Delta u > 0$ in a neigbourhood of 0, and so the integration formula for φ' shows that $\varphi'(\rho) > 0$ for sufficiently small ρ. Therefore, for every sufficiently small $\rho > 0$, $\varphi(\rho) > \varphi(0) = u(0)$. But since $\varphi(\rho)$ is the average of u over $\partial B(0, \rho)$, there must be some points $y \in \partial B(0, \rho)$ with $u(y) > u(0)$, and so 0 cannot be a local maximum (this argument works just as well with any point x replacing 0, of course).

#3.5: If $u > 1$ somewhere in Ω, u will achieve its maximum at some point $x \in \Omega$ — compactness guarantees it achieves its maximum in $\overline{\Omega}$, but it cannot happen on the boundary because $u = 0$ there. At this point we get $\Delta u(x) = u(x)^3 - u(x) > 0$ which is impossible (see #3.4 above). We can argue similarly about the case when $u < -1$ somewhere, but it is easier to just replace u by $-u$ and use what we already proved, since $-u$ satisfies the same conditions as u does.

#3.6: In the "more difficult" proof of #3.4 above, we have in fact proved the assertion

$$v(x) \le \fint_{\partial B(x,r)} v \, \mathrm{d}S(y)$$

for subharmonic v (at least if you replace 0 by x and the strict inequalities by nonstrict ones.) To get the desired inequality is essentially a question of integrating in polar coordinates:

$$\begin{aligned}
\fint_{B(x,r)} v(y)\,\mathrm{d}y &= \frac{1}{\alpha(n)r^n}\int_{B(0,r)} v(x+y)\,\mathrm{d}y = \frac{1}{\alpha(n)r^n}\int_0^r \int_{\partial B(0,\rho)} v(x+y)\,\mathrm{d}S(y)\,\mathrm{d}\rho \\
&= \frac{1}{\alpha(n)r^n}\int_0^r n\alpha(n)\rho^{n-1} \underbrace{\fint_{\partial B(0,\rho)} v(x+y)\,\mathrm{d}S(y)}_{\ge v(x)}\,\mathrm{d}\rho \\
&\ge \frac{v(x)}{r^n}\int_0^r n\rho^{n-1}\,\mathrm{d}\rho = v(x)
\end{aligned}$$

If v has a maximum point in $x \in U$, the above inequality will imply that $v(y) = v(x)$ for all y in a neighbourhood of x: For when $v(y) \le v(x)$ for $y \in B(x,r)$ then instead of the above inequality we find the opposite inequality — and it becomes strict if $v(y) < v(x)$ for some y. Then the same connectedness argument as in the proof of the strong maximum princliple shows that u is constant in the component of U containing x, and so u achieves its maximum on the boundary of that component.

To show $\varphi(u)$ is subharmonic when φ is convex and u is harmonic, is simple: Just compute $\varphi(u)_{x_i x_i} = \varphi''(u)u_{x_i}^2 + \varphi'(u)u_{x_i x_i}$ and add them all together to get $\Delta\varphi(u) = \varphi''(u)|Du|^2 + \varphi'(u)\Delta u \ge 0$ since the first term is nonnegative and the second is zero.

Finally, if u is harmonic then so is u_{x_i}, and therefore $u_{x_i}^2$ is subharmonic by the previous result. Add them all together to get $|Du|^2$ subharmonic.

Problem set #4 — for next week

#4.1: Let Ω be an open connected set, and let (u_n) be a sequence of positive harmonic functions on Ω. Assume that $u_n(x) \to 0$ for some $x \in \Omega$. Show that then $u \to 0$ uniformly on compacts [i.e., $u_n|_K \to 0$ uniformly for each compact $K \subset \Omega$].

#4.2: Use the idea from the proof of Harnack's inequality to prove Liouville's theorem: A bounded, harmonic function on $\mathbb{R}^n$ is constant. (*Hint*: You may assume that $u \ge 0$ (why?). Fix two points x and y in $\mathbb{R}^n$ and set $a = |x - y|$. Use the inclusion $B(y,r) \subset B(x, r+a)$ to compare $u(x)$ with $u(y)$ and let $r \to \infty$.) This actually proves a stronger result than Liouville's theorem. How much extra can you get out of this proof?

#4.3: Find a nonzero solution to the Dirichlet problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ where Ω is the strip $\{(x,y) \in \mathbb{R}^2 \colon 0 < y < \pi\}$. (*Hint*: Use separation of variables.) What does this example tell you about the maximum principle?

#4.4: Evans problem 6 (p. 81).

#4.5: Assume $\Omega \subset \mathbb{R}^n$ is open and bounded, and assume $u, v \in C(\overline{\Omega})$ with u subharmonic and v harmonic in Ω. If $u \le v$ on $\partial\Omega$, show that $u \le v$ in Ω (this is the reason for the term *subharmonic*).

#4.6: Prove the Hadamard "Three Circles" theorem: Let $u(x,y)$ be subharmonic in a domain containing the annulus $r_1^2 \le x^2 + y^2 \le r_2^2$. Then the maximum

$$M(r) = \max_{x^2+y^2=r^2} u(x,y)$$

of u on any concentric circle satisifies

$$M(r) \le \frac{M(r_1)\ln\dfrac{r_2}{r} + M(r_2)\ln\dfrac{r}{r_1}}{\ln\dfrac{r_2}{r_1}}$$

where $r_1 < r < r_2$. (*Hint*: Any function of the form $a\ln r + b$ is harmonic in $\mathbb{R}^2$. Compare using suitable choices of a and b.)

(*Remark.* In the corresponding "Three Spheres" theorem in $\mathbb{R}^n$ the quantity $\ln(b/a)$ is replaced by $a^{2-n} - b^{2-n}$.)