75042 Partial differential equations

Problem set #1 — suggested solutions

The problems this week were picked from the notes entitled "Elements of analysis".

Number 2: To show that $B_\varepsilon(x)$ is open, let $y \in B_\varepsilon(x)$. Let $\delta = \varepsilon - d(x,y)$. Then $\delta > 0$, and $B_\delta(y) \subseteq B_\varepsilon(x)$, since $d(y,z) < \delta \Rightarrow d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta = \varepsilon$.

Next, let A be a set and A° its interior. If $x \in A^\circ$ then by definition there is some $\varepsilon > 0$ so $B_\varepsilon(x) \subseteq A$. However if $y \in B_\varepsilon(x)$ then some δ–neighbourhood of y is contained in $B_\varepsilon(x)$ and therefore in A. Thus A is a neighbourhood of y. In other words, $y \in A^\circ$. This shows that A° is open.

Finally, if $y \in X \setminus \bar{B}_\varepsilon(x)$ let $\delta = d(x,y) - \varepsilon$. Then $\delta > 0$ and $B_\delta(y) \subseteq X \setminus \bar{B}_\varepsilon(x)$, for if $d(y,z) < \delta$ and $z \in \bar{B}_\varepsilon(x)$ then $d(x,y) \leq d(x,z) + d(z,y) < \varepsilon + \delta = d(x,y)$ which is impossible.

Number 3: Assume the sequence (x_n) has the two limits y and z. By the triangle inequality $d(y,z) \leq d(y,x_n) + d(x_n,z)$ for any n. Given any $\varepsilon > 0$ we can make sure that $d(x_n,y) < \varepsilon$ and $d(x_n,z) < \varepsilon$ for n large enough. If we pick $\varepsilon = d(y,z)/2$ a contradiction results.

Number 4: Assume the subsequence (y_k) given by $y_k = x_{n_k}$ has the limit x. For any given $\varepsilon > 0$, there exists some K so that $d(y_k,x) < \varepsilon$ whenever $k \geq K$. Given any N, we find some K' so that $n_k \geq N$ whenever $k \geq K'$. Whenever $k \geq \max(K,K')$ then $n_k \geq N$ and $d(x_{n_k},x) < \varepsilon$. This shows that x is a limit point for the sequence (x_n).

On the other hand, if x is a limit point for (x_n) then pick any sequence of real numbers $\varepsilon_k > 0$ with $\varepsilon_k \searrow 0$ (by this we mean that the sequence (ε_k) is nonincreasing and converges to 0). Let n_1 be so that $d(x_{n_1},x) < \varepsilon_1$, and for $k = 2,3,\ldots$ let $n_k > n_{k-1}$ be so that $d(x_{n_k},x) < \varepsilon_k$ (the existence of these is exactly the definition of limit point). Now $x_{n_k} \to x$, for if $\varepsilon > 0$, we find first some K with $\varepsilon_K < \varepsilon$; and then, whenever $k \geq K$, we have $d(x_{n_k},x) < \varepsilon_k \leq \varepsilon_K < \varepsilon$.

Number 5: We prove $(a) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

$(a) \Rightarrow (d)$: Assume F is closed. Given a sequence in F with a limit point x, that sequence has a subsequence converging to x. Then $x \in F$ because F is closed.

$(d) \Rightarrow (c)$: Assume F contains every limit point of every sequence in F. Assume $x \in X$ is such that every neighbourhood of x meets F (we say *A meets B* if $A \cap B \neq \emptyset$). Let $\varepsilon_n > 0$ with $\varepsilon_n \searrow 0$. For each n, let $x_n \in F \cap B_{\varepsilon_n}(x)$. Then $x_n \to x$, and so $x \in F$ because x is a limit point of the sequence (x_n).

$(c) \Rightarrow (b)$: Let $x \in X \setminus F$. If (c) holds, then there must exist a neighbourhood of x which does not meet F. This neighbourhood, then, is contained in $X \setminus F$ which is itself, therefore, a neighbourhood of x. Thus $X \setminus F$ is a neighbourhood of all its points, and is therefore open.

$(b) \Rightarrow (a)$: Assume $X \setminus F$ is open and let (x_n) be a sequence in F, $x_n \to x$. If $x \notin F$ then $x \in X \setminus F$, so $X \setminus F$ is a neighbourhood of x. This neighbourhood must then contain a tail of the sequence (x_n), which is impossible since $x_n \in F$ for all n.

Number 12: For all n, we have $\inf\{x_k : k \geq n\} \leq \sup\{x_k : k \geq n\}$. Taking limits in this inequality immediately yields $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$.

Assume $L = \overline{\lim}\, x_n$ is finite. Since $S_n = \sup\{x_k : k \geq n\} \searrow L$, then given $\varepsilon > 0$ there is some N so that $L \leq S_n < L + \varepsilon$ whenever $n \geq N$. By the definition of the supremum there is some $k \geq n$ so that $S_n - \varepsilon < x_k \leq S_n$. For any such k, then $|x_k - L| \leq |x_k - S_n| + |S_n - L| < \varepsilon + \varepsilon = 2\varepsilon$. Thus L is a limit point for (x_n).

No limit point x exists with $x > \overline{\lim}\, x_n$, for if this inequality holds then for some sufficiently large n, $\sup\{x_k : k \geq n\} < x$. Then this tail of the sequence does not meet $B_\varepsilon(x)$ where $\varepsilon = x - \sup\{x_k : k \geq n\}$, so x is not a limit point.

Similarly to the above two results $\underline{\lim}\, x_n$ is a limit point, and no limit point x with $x < \underline{\lim}\, x_n$ can exist. This is most easily seen from the previous results and the identity $\underline{\lim}\, x_n = -\overline{\lim}(-x_n)$ (or you can go through the proofs, reversing inequalities).

If $x_n \to x \in \mathbb{R}$ and $\varepsilon > 0$, we get in the usual way $x - \varepsilon < x_n < x + \varepsilon$ whenever $n \geq N$ — for some N. Then $x - \varepsilon \leq \inf\{x_n : n \geq N\} \leq \sup\{x_n : n \geq N\} \leq x + \varepsilon$. Taking limits, we get $x - \varepsilon \leq \underline{\lim}\, x_n \leq \overline{\lim}\, x_n \leq x + \varepsilon$. Since this holds for any $\varepsilon > 0$, we must have $\underline{\lim}\, x_n = \overline{\lim}\, x_n = x$.

On the other hand, if $\underline{\lim}\, x_n = \overline{\lim}\, x_n = x \in \mathbb{R}$, then whenever $\varepsilon > 0$ we have $x - \varepsilon < \inf\{x_n : n \geq N\} \leq \sup\{x_n : n \geq N\} < x + \varepsilon$ for sufficiently large N, and so whenever $n \geq N$, $x - \varepsilon < x_n < x + \varepsilon$. In other words $|x_n - x| < \varepsilon$, so $x_n \to x$.

Number 13: It seems to me that the only unproved points in the proof are that Cauchy sequences are bounded (but that is too trivial to bother with here?) and that any limit point of a Cauchy sequence is in fact the limit of that sequence. So to prove the latter, let X be any metric space (complete or not), and let (x_n) be a Cauchy sequence in X with a limit point x. To show that $x_n \to x$, let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there is some N so that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$. But since x is a limit point, there is at least one point x_m with $m \geq N$ so that $d(x_m, x) < \varepsilon$. But then, given *any* $n \geq N$ we have $d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) < \varepsilon + \varepsilon = 2\varepsilon$. And so we have proved $x_n \to x$.

Number 15: Let K be a closed and bounded subset of $\mathbb{R}$ and let (x_n) be a sequence in K. Since K is bounded, so is (x_n), and by Theorem 14, that sequence has at least one limit point in $\mathbb{R}$. Since K is closed, that limit point is in K. This proves that K is compact.

Number 16: Let X be a compact metric space and K a closed subset of X. If (x_n) is a sequence in K then it has a limit point in X, and since K is closed, that limit point belongs to K.

Alternative proof: Let $\mathscr{V}$ be an open cover of K in the following sense: Each member of $\mathscr{V}$ is an open subset of X, and their union contains K. Then $\mathscr{V} \cup \{X \setminus K\}$ is an open cover of X, and so has a finite subcover $\mathscr{U}$ since X is compact. This will also be a finite cover of K. This proves that K is compact provided the following is clear:

> Let X be a metric space and Y any subset of X. Then, as remarked in the notes, Y is itself a metric space. An open subset of Y is defined in the usual way. It turns out that a subset $V \subseteq Y$ is open in Y if and only if we can write $V = Y \cap U$ where $U \subseteq X$ is open in X. In fact, for each $v \in V$ we can find $\varepsilon(v) > 0$ so that $y \in V$ whenever $y \in Y$ and $d(y, v) < \varepsilon(v)$. Let $U = \bigcup_{v \in V} B_{\varepsilon(v)}(v)$ (a union of balls in X). Then U is open, and $Y \cap U = V$.

Number 23: Consider the triangle inequality $d(x, a) \leq d(x, y) + d(y, a)$. With $a \in A$ this implies $\operatorname{dist}(x, A) \leq d(x, y) + d(y, a)$. Since this holds for all a, we can take the infimum to get $\operatorname{dist}(x, A) \leq d(x, y) + \operatorname{dist}(y, A)$. In other words we have $\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq d(x, y)$. The same inequality with x and y interchanged yields $|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y)$, which clearly shows that $\operatorname{dist}(x, A)$ is a continuous function of x.

If $K \cap \bar{A} = \emptyset$, then the continuous function $x \mapsto \operatorname{dist}(x, A)$ is positive on K, for $\operatorname{dist}(x, A) > 0$ if and only if $x \notin \bar{A}$. If K is compact, that function achieves its minimum on K, and so there is a constant $\varepsilon > 0$ so that $\operatorname{dist}(x, A) \geq \varepsilon$ for each $x \in K$.

Problem set #2 — for next week

The problems this week were picked from the notes entitled "Elements of analysis": Problems number **24** and **27**.

What is the Laplacian in polar coordinates (in two variables)? Use separation of variables to express a solution of the boundary value problem $\Delta u = 0$ in $x^2 + y^2 < 1$, $u(e^{i\theta}) = g(\theta)$ in terms of the Fourier series of g. Separation of variables should lead to solutions of the form $r^{|n|}e^{in\theta}$ for $n = 0, \pm 1, \pm 2, \ldots$. After matching the boundary data with the Fourier series for g using the explicit formula for the Fourier coefficients, interchange the order of summation and integration. Note that the resulting sum is a geometric series which can be summed explicitly. The resulting integral representation is the famous *Poisson's integral formula*. Can you show rigorously that the resulting function u really solves $\Delta u = 0$? How about the boundary condition? (The latter is quite hard if you haven't seen it before — but we will get back to it.)