# TMA4195 Mathematical modelling 2005

## *Exercise set 8*

Advice and suggestions: 2005–11–02

*This exercise is to be handed in by Friday, 11 November.*
It will count as 10 % toward your final grade.

We wish to use a laser to drill holes in various materials. The laser has a total output effect $P$, and illuminates a small area $A$ with uniform intensity $I = P/A$. The energy of the laser beam is used to heat the material from the ambient temperature $T_0$ to the boiling point $T_b$, and then evaporate it. We ignore the phase transition from solid to liquid, and the corresponding latent heat of melting.

**(a)** Give an argument showing that if the depth of the hole at time $t$ is $s$, then

$$\frac{ds}{dt} = \frac{I}{\big(\lambda + c(T_b - T_0)\big)\rho}$$

provided we ignore heat conduction into the material.
(See the table at the end for the meaning of some of the constants.)

We will now take heat conduction into account, but in one dimesion only (parallell with the hole and the laser beam). This should be a good approximation provided the heat does not penetrate far into the material compared with the diameter of the hole. When the laser beam is switched on ($t = 0$) the material has uniform temperature $T_0$, and the depth of the hole is $s(0) = 0$. Let the $x$ axis point into the material, so that the bottom of the hole always satisfies $x = s(t)$.

We shall assume that heat conduction in the material is given by the heat equation

$$c\rho T_t = kT_{xx}$$

A certain time period $t_b$ will lapse before the surface is heated to the boiling point $T_b$ and the hole begins to form.

**(b)** Give an argument showing that $T(x,t)$ satisfies

$$\begin{aligned} T(x,0) &= T_0 && (x > 0) \\ kT_x(0,t) &= -I && (0 < t < t_b) \\ T\big(s(t),t\big) &= T_b && (t_b < t) \end{aligned}$$

**(c)** Set up an energy budget for the energy in the control volume $s(t) < x < x_1$ (where $x_1$ is a fixed point) which accounts for all forms of energy transportation, and use this to show that, when $t > t_b$,

$$\lambda\rho\frac{ds}{dt} = I + kT_x\big(s(t),t\big)$$

**(d)** After sufficient time has elapsed, it seems probable that the temperature profile in the material is approximately constant, i.e.,

$$T(x,t) = T_\ell\big(x - s(t)\big)$$

for some function $T_\ell$. Show that in this case $ds/dt$ is given by the same formula as in question (a).

**(e)** Explain why the only parameters of importance for solving the problem are the combinations $I$, $T_b - T_0$, $k$, $\lambda\rho$, and $c\rho$. Show that the only dimensionless combination of those is

$$\varepsilon = \frac{c\rho(T_b - T_0)}{\lambda\rho} = \frac{c(T_b - T_0)}{\lambda}$$

and powers thereof.
For many materials of interest (aluminium, iron etc.) $\varepsilon \approx 0.2$, so $\varepsilon$ may be considered a small parameter.

**(f)** Suggest a suitable scaling for the problem, and write up the scaled equations.

**(g)** Find the most general solution to the scaled equations – except the initial conditions – which has a constant profile (like in question (d) – why the exception?).

*We may return to the solution for small times in a later problem set.*

**Appendix.** Some physical dimensions and their units:

| | | |
|---|---|---|
| Energy | $E$ | $\mathrm{J} = \mathrm{kg\,m^2 s^{-2}}$ |
| Power | $P$ | $\mathrm{W} = \mathrm{kg\,m^2 s^{-3}}$ |
| Heat conductivity | $k$ | $\mathrm{W K^{-1} m^{-1}}$ |
| Specific heat capacity | $c$ | $\mathrm{J\,kg^{-1} K^{-1}}$ |
| Density | $\rho$ | $\mathrm{kg\,m^{-3}}$ |
| Heat of evaporation | $\lambda$ | $\mathrm{J\,kg^{-1}}$ |