Suggested solution Mathematical modelling
Exercise 6, autumn 2005

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Exercise 1 – A simple expansion near a singularity

We have been given the function

$$f(x;\epsilon) = \frac{1}{x+\epsilon}. \tag{1}$$

First we expand it using the binomial theorem. The binomial theorem is given by

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k a^{-n-k},$$

we use it to expand around ϵ and end up with

$$f(x;\epsilon) \approx \frac{1}{x} - \frac{\epsilon}{x^2} + \cdots$$

as expected.

We now rescale using $x = \epsilon X$, yielding

$$F(X;\epsilon) = \frac{1}{\epsilon(1+X)}.$$

Again we expand this into series, ending up with

$$F(X;\epsilon) = \frac{1}{\epsilon}\left(1 - X + X^2 - \cdots\right) = \frac{1}{\epsilon} - \frac{x}{\epsilon^2} + \frac{x^2}{\epsilon^3} - \cdots$$

which is clearly valid for $|X| < 1$, and thus for $|x| < \epsilon$.

In fact, this is nothing but what we would get from the first procedure if we interchanged x and ϵ.

Exercise 4 – A two-point boundary value problem

This time our starting point is the equation

$$\begin{aligned} &\epsilon \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad 1 < x < 2 \\ &y(1) = 0,\ y(2) = 1 \qquad 0 < \epsilon \ll 1. \end{aligned} \tag{2}$$

The first thing we do is to expand and obtain the outer solution. We let

$$y = y_0 + \epsilon y_1 + \cdots$$

and insert into (2). A 0'th order solution is given by

$$x\frac{dy_0}{dx} + y_0 = 0.$$

This is a separable equation with the solution

$$y_0 = \frac{C_1}{x}.$$

So how can we guess the location of the boundary layer? The boundary layer – wherever it may be – is recognized by the fact that y'' is large there. So y' changes rapidly. Inside the boundary layer, then we can treat the x in the equation as an approximate constant. So no the approximate equation within the boundary layer (near x_0) is

$$\epsilon y'' + x_0 y' + y = 0,$$

whose general solution is $Ae^{r_1 x} + Be^{r_2 x}$ where r_1, r_2 are the roots of the equation $\epsilon r^2 + x_0 r + 1 = 0$, i.e., $r = \frac{1}{2}\epsilon^{-1}(-x_0 \pm \sqrt{x_0^2 - 4\epsilon}$. These roots are approximately $r_1 \approx -1/x_0$ and $r_2 \approx -x_0/\epsilon$. The solution corresponding to the latter root will go rapidly to 0 when x grows (if $\epsilon > 0$) or when x decreases (if $\epsilon < 0$).[1] Therefore we expect a boundary layer at the left end of the interval when $\epsilon > 0$, and at the right end if $\epsilon < 0$.

Here $\epsilon > 0$, so we proceed assuming a boundary layer at the left end. We use the boundary condition at $x = 2$ and end up with

$$y_0 = \frac{2}{x}.$$

Then we look for the inner solution. We let

$$x = 1 + \epsilon X,$$

insert and expand $Y = Y_0 + \epsilon Y_1 + \cdots$ where $y(x) = Y(X)$. A 0'th order solution is given by

$$\frac{d^2 Y_0}{dX^2} + \frac{dY_0}{dX} = 0,$$

with the solution $Y_0(X) = C_1 + C_2 e^{-X}$. We have $Y(0) = 0$ which yields $C_1 + C_2 = 1$. We then match the inner and outer solution, that is we require that $\lim_{X\to\infty} Y(X) = \lim_{x\to 1} y(x)$, or $C_1 = 2$. Our final inner solution is given by

$$Y_0(X) = 2 - 2e^{-X},$$

and a reasonable approximation to the full solution is

$$y(x) + Y(X) - 2 = y(x) + Y\left(\frac{x-1}{\epsilon}\right) - 2 = \frac{2}{x} - 2e^{(1-x)/\epsilon}.$$

As to how this fits together with the van Dyke matching rule, consider this:

Start with the 1-term outer expansion $y_0 = 2/x$. In inner variables it becomese $y_0 = 2/(1+\epsilon X)$. Express that as a power series: $y_0 = 2(1 - \epsilon X + \cdots$. Keep just one term: $y_0 \sim 2$.

Next, start with the 1-term inner expansion $Y_0 = C_1(1 - e^{-X})$. Express it in the outer variable: $Y_0 = C_1(1 - e^{(1-x)/\epsilon})$ and note that if $x > 1$ then $e^{(1-x)/\epsilon}$ is exponentially small as $\epsilon \to 0$. So the 1-term outer expansion becomes simply $Y_0 \sim C_1$.

Thus the van Dyke matching rule is just $C_1 = 2$, as for the simpler matching method.

[1] I am also using $x_0 > 0$ here.

Exercise 5 – An artificial example

As always, we start with an equation, this time given by

$$\epsilon \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u + u^3}{1 + 3u^2}, \qquad u(0) = 0, \quad u(1) = 1. \tag{3}$$

We obtain the outer solution in the usual way. Start by expanding $u = u_0 + \epsilon\, u_1 + \cdots$, then insert this into (3) and look at the 0'th order solution. It is given by

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = \frac{u_0 + u_0^3}{1 + 3u_0^2}$$

which has the general solution $u_0 + u_0^3 = C_1 e^x$. Using the right boundary condition we get

$$u_0 + u_0^3 = 2e^{x-1}.$$

We then look for the inner solution, by letting $x = \epsilon\, X$ and expand $U = U_0 + \epsilon\, U_1 + \cdots$ where $u(x) = U(X)$. We get the ODE

$$\frac{\mathrm{d}^2 U_0}{\mathrm{d}^2 X} + \frac{\mathrm{d}U_0}{\mathrm{d}X} = 0$$

with general solution $U_0(X) = C_1 + C_2 e^{-X}$. We have $U(0) = 0$ which gives $C_1 + C_2 = 0$, so that

$$U_0(X) = C_1(1 - e^{-X}).$$

We match the solutions by demanding that $\lim_{X\to\infty} C_1(1 - e^{-X}) = u_0(0)$, i.e., $C_1 = u_0(0)$, which we can determine by $C_1 + C_1^3 = 2e^{-1}$.

Exercise 10 – Logarithms

We are asked to show that the one-term outer expansion of

$$f(x;\epsilon) = 1 + \frac{\log x}{\log \epsilon}, \quad x > 0 \tag{4}$$

is given by $f \sim 1$. This is really nothing much more that the statement that $f(x;\epsilon) \to 1$ as $\epsilon \to 0$, which is obviously true.

In the boundary layer near $x = 0$ we then let $x = \epsilon\, X$, and write $f(x;\epsilon) = F(X;\epsilon)$. We get

$$F(X;\epsilon) = 1 + \frac{\log \epsilon X}{\log \epsilon} = 1 + \frac{\log \epsilon}{\log \epsilon} + \frac{\log X}{\log \epsilon} = 2 + \frac{\log X}{\log \epsilon}$$

which yields $F \sim 2$ to the first order. Obviously there is no way to match these two solutions.

If we include the logarithmic term, however, we trivially have $F(X;\epsilon)$ exactly in the inner expansion, and $f(x;\epsilon)$ exactly in the outer expansion. These two match because they are in fact equal by definition.