Suggested solution Mathematical modelling
Exercise 3, autumn 2005

Arne Morten Kvarving / Harald Hanche-Olsen

October 4, 2005

Exercise 1

We use Buckingham's Pi theorem directly:

$$\begin{aligned}
[D] &= [\mathrm{m}]\,, & [\rho] &= \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \\
[h] &= [\mathrm{m}]\,, & [d] &= [\mathrm{m}]\,, \\
[E] &= \left[\frac{\mathrm{kg}}{\mathrm{m\,s}^2}\right], & [\delta] &= [\mathrm{m}]\,, \\
[g] &= \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].
\end{aligned}$$

From these 7 quantities involving 3 dimensions, we excpect to find $7-3=4$ dimensionless quantities, for example[1]

$$\begin{aligned}
\pi_1 &= \frac{\delta}{D}, & \pi_2 &= \frac{h}{D} \\
\pi_3 &= \frac{d}{D}, & \pi_4 &= \frac{E}{Dg\rho}.
\end{aligned}$$

According to Buckingham's theorem, the wanted relation can be written

$$\frac{\delta}{D} = \phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{Dg\rho}\right).$$

Obviously, the bottom of the tank sags because of the pressure of the water, which is given as $p = \rho g h$ on the bottom of the tank. We can get rid all occurences of rho, g and h except for the combination $\rho g h$ by replacing π_2 and π_4 by π_4/π_2, leading to

$$\frac{\delta}{D} = \phi\left(\frac{d}{D}, \frac{E}{hg\rho}\right).$$

Of course, we could have arrived at this directly if we had the good foresight to replace D by h in the definition of π_4. Even better, we could have started out with the five relevant variables D, p, d, E, δ and saved a bit of work.

[1] The first three are sort of obvious, and π_4 can also be found by inspection: To include E in a dimensionless combination, we need to divide by ρ in order to get rid of all kilograms. Then E/ρ needs to be divided by g to get rid of the seconds, and we're left with a length, so we divide by one of the length parameters.

Exercise 2

We start out with the quantities and their dimensions:

$$[V] = \left[\mathrm{m}^3\right], \qquad [\rho] = \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \quad [t] = [\mathrm{s}],$$
$$[P] = \left[\frac{\mathrm{kg}}{\mathrm{m\,s}^2}\right], \quad [\mu] = \left[\frac{\mathrm{kg}}{\mathrm{m\,s}}\right].$$

From these 5 quantities expressed in 3 dimensions, we expect to extract $5-3=2$ independent dimensionless quantities. There are many ways to do this, but for the best correspondence with the original P vs t graphs we should look for two combinations where one contains P but not t, and the other contains t but not P. This produces

$$\pi_1 = \frac{\mu t}{V^{2/3}\rho}, \quad \pi_2 = \frac{V^{2/3}\rho P}{\mu^2}.$$

Buckingham's theorem states that, if t is a function of the other parameters, this can be written as $\pi_1 = \phi(\pi_2)$ – so the axes in the unified graph should be π_2 (horizontal) and π_1 (vertical).

Oppgave 3

For any of these materials, use the thermal diffusion coefficient $\kappa = \frac{k}{\rho c}$. From our work with the heat equation we know that $[\kappa] = [\mathrm{m}^2/\mathrm{s}]$, and that heat diffuses a distance $\delta \sim \sqrt{\kappa t}$ over a time t.

For simplicity, assume a homogeneous fish consisting of all water, fat or protein. Pick $t = 30\,\mathrm{s}$ according to the design criteria for the instrument.

The different substances produces these δ values:

$$\delta_{\text{water}} \approx 2.0\,\mathrm{mm}$$
$$\delta_{\text{fat}} \approx 1.2\,\mathrm{mm}$$
$$\delta_{\text{protein}} \approx 1.9\,\mathrm{mm}.$$

Thus it appears that the heat does not penetrate deeply enough to provide any information beyond the first two millimeters or so.