Suggested solution Mathematical modelling
Exercise 2, autumn 2005

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Exercise 3 – The Boussinesq transformation:

We have been given the equation

$$\mathrm{Pe}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\nabla^2 T. \tag{1}$$

The first thing we need to do is to find the partial derivatives expressed using ϕ and ψ:

$$\frac{\partial}{\partial x}=u\frac{\partial}{\partial\phi}-v\frac{\partial}{\partial\psi}, \tag{2}$$

$$\frac{\partial}{\partial y}=v\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial\psi}, \tag{3}$$

$$\frac{\partial^2}{\partial x^2}=u^2\frac{\partial^2}{\partial\phi^2}-2uv\frac{\partial}{\partial\phi}\frac{\partial}{\partial\psi}+v^2\frac{\partial^2}{\partial\psi^2}, \tag{4}$$

$$\frac{\partial^2}{\partial y^2}=v^2\frac{\partial^2}{\partial\phi^2}+2uv\frac{\partial}{\partial\phi}\frac{\partial}{\partial\psi}+u^2\frac{\partial^2}{\partial\psi^2}. \tag{5}$$

We then insert (2)-(5) into (1) and do some rearrangements. This yields

$$\begin{gathered}\mathrm{Pe}\left(u^2+v^2\right)\frac{\partial T}{\partial\phi}=\left(u^2+v^2\right)\left(\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial\psi^2}\right)\\ \Updownarrow\\ \mathrm{Pe}\frac{\partial T}{\partial\phi}=\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial\psi^2}.\end{gathered} \tag{6}$$

We also need to find the boundary conditions expressed in our new variables. The stream function ϕ is given in polar coordinates at p.p. 29 in the book. In carthesian coordinates it reads

$$\phi=U\left(x+\frac{a^2x}{r^2}\right).$$

But we need $w(z)=\phi+\mathrm{i}\psi$ to be an analytic function. With $z=x+iy$ and $r=|z|$ this fits with the function

$$w(z)=\phi+\mathrm{i}\psi=U\left(z+\frac{a^2}{z}\right).$$

From this we see that we have

$$\phi=U\left(1+\frac{a^2}{r^2}\right)x,\quad \psi=U\left(1-\frac{a^2}{r^2}\right)y.$$

The boundary conditions are thus given by

$$\begin{aligned} T &= T_0, \qquad |\phi| \leqslant 2Ua, \quad \psi = 0 \\ T &= T_\infty, \qquad |\phi| + |\psi| \to \infty \end{aligned} \tag{7}$$

Exercise 5 – Newton's law of cooling and Biot numbers

This time we are going to have a look at Newton's law of cooling, expressed mathematically as

$$-k\,\frac{\partial T}{\partial n}\bigg|_{\text{boundary}} = h\,(T - T_\infty)\,. \tag{8}$$

Simple analysis of the units on the left hand side shows that

$$[h] = \left[\frac{\mathrm{J}}{\mathrm{m}^2\,\mathrm{s}\,\mathrm{K}}\right] = \left[\frac{\mathrm{kg}}{\mathrm{K}\,\mathrm{s}^3}\right].$$

That energy will leak out of the surface of the body should be obvious (this explains the negative sign). That this energy flux is proportional with the temperature difference between the body and the surroundings is also quite plausible.

Since Stefan–Boltzman's law describes a heat flux the units of K must be

$$[K] = \left[\frac{\mathrm{J}}{\mathrm{s}\,\mathrm{K}^4}\right] = \left[\frac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{K}^4\,\mathrm{s}^3}\right]. \tag{9}$$

To show that Newton's law is a good approximation to Stefan–Boltzman's law we use the secant law. ** HVA HETER DENNE? ** It says that

$$KT^4 - KT_\infty^4 = 4K(T^*)^3\,(T - T_\infty) \tag{10}$$

for some $T_\infty \leqslant T^* \leqslant T$. As long as $T - T_\infty$ is acceptably small we can assume that this secant is a good approximation to the actual curve. If we compare this to Newton's law we find that

$$h = \frac{4K(T^*)^3}{S} \approx \frac{4K(T_\infty)^3}{S} \tag{11}$$

where S is the surface area of the body.

The first thing we do when we want to write the problem in dimensionless form is to use Buckingham's Pi theorem to find the dimensionless combinations of the parameters in the problem. We have $n = 4$ and $r = 4$ so we expect to find $k = 1$ dimensionless combinations. The single combination is given by

$$\mathrm{Bi} = \frac{hL}{k}.$$

We let $T' = \frac{T - T_\infty}{T_0}$, $x = Lx'$ and $y = Ly'$. With these scalings we end up with

$$-\frac{kT_0}{L}\frac{\partial T'}{\partial n'} = hT_0T',$$

which fits with the form given in the exercise. (This means that we could have discovered the Biot number directly from the equation, without doing any dimensional analysis.)

Exercise 6 – Coffee time

Here we have to do some assumptions. The first assumptions we make is that we can use Newton's law of cooling, and that all energy leak through the boundary between the coffee and the air (e.g. we have a thermo-cup). We let

- V_m = volume of the milk.
- V_k = volume of the coffee.
- S = surface area of the coffee – assumed constant.
- c_k = specific heat capacity of the coffee – we assume that this is approximately equal for a mixture of coffee and milk.
- c_m = specific heat capacity of the milk.
- T_∞ = room temperature – assumed constant in the entire house.
- T_k = temperature of the coffee when it is obtained from the machine.
- T_m = temperature of the milk when it is poured into the coffee – we assume $T_m < T_\infty$.
- τ = the time it takes to walk back to the office.

If we let $T' = T - T_\infty$ the temperature change is given by

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = \frac{-h}{c_k} T' S.$$

This means that

$$T' = T'_0 e^{-hSt/c_k}. \tag{12}$$

The temperature change in the coffee due to the addition of the milk is given by

$$\Delta T = \frac{(T - T_m)\, c_m V_m}{c_k V_k}. \tag{13}$$

We can then express the temperature change if he adds the milk at the machine as $\Delta T = \left(T_k - T_m\right) c_m V_m / c_k V_k$ som gir $T'_0 = T_k - \Delta T - T_\infty$. If we insert this into the solution for T' we get

$$T' = (T_k - \Delta T - T_\infty) e^{-hS\tau/c_k}.$$

However, if the milk is added at the office we have $T'_0 = T_k - T_\infty$. We then substract ΔT resulting in

$$T' = \frac{(T_k - T_\infty) e^{-hS\tau/c_k} - \left(T' + T_\infty - T_m\right)\alpha}{1 + \alpha}$$

where $\alpha = c_m V_m / (c_k V_k)$. Since $\alpha > 0$ and $T_\infty > T_m$ (assumed) the model indicates that it is preferable to add the milk at the machine.

The amount of sugar that is dissolved should obviously depend on the area that is in contact with the coffee. One possibility is that the volume change is proportional to this area, as suggested in the exercise text.

Assuming that these two are the only two parameters in the problem, there is only one way to put together a dimensionless combination, namely $\pi_1 = A/V^{2/3}$. This immediately shows us that $A \propto V^{2/3}$.

We are then asked to solve the model given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -C_1 V^{2/3}.$$

This is a separable ordinary differential equation, after the integration we end up with

$$V(t) = \frac{1}{3}\left(t_0 - C_1 t\right)^3 . \tag{14}$$

From this we see that $V = 0$ for $t = T_0$, that is, the volume reaches zero in finite time.

We are then asked to solve the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = V^{2/3} \qquad t > 0 \tag{15}$$

with initial condition $V(0) = 0$. A trivial solution is $V(t) = 0 \ \forall \ t \geq 0$. But this is not the only solution! Since the requirements for an unique solution of the initial value problem fail to be fulfilled. That is, the function $f(V) = V^{2/3}$ is not Lipschitz – there is no constant L such that $|f(V) - f(V')| \leq L|V - V'|$ for all V, V'. But it is Lipschitz in any interval $[\delta, \infty)$ where $\delta > 0$.

If $V \neq 0$ we can solve this as a separable ordinary differential equation, resulting in $3V^{1/3} = t - t_0$, that is

$$V = \frac{1}{27}(t - t_0)^3.$$

The solution is zero for $t = t_0$, and this concides with $V = 0$. Thus we can not prove uniqueness of the solutions. If you think about it you should be able to convince yourself that you have to “connect” the trivial solution and the one we just found. The general solution to the given initial value problem is given by

$$V = \begin{cases} 0 & 0 \leq t \leq t_0 \\ \frac{1}{27}(t - t_0)^3 & t > t_0 \end{cases}$$

where $0 \leq t_0 \leq \infty$ (the trivial solution is this solution with $t_0 = \infty$).

To figure out when a sugar lump was added to the coffee based on its current state, we have to solve the given differential equation with an initial value (actually a final value) $V(t_{\text{observed}}) = V_{\text{observed}}$, and then find the value for $V(0)$.

What we are doing is reversing the time, we use $t' = t_{\text{observed}} - t$ as our independent variable. This concides with solving the differential equation we just solved. As we have seen, there is no unique solution if $V_{\text{observed}} = 0$.

A physical interpretation of this is that when the sugar lump is completely dissolved, we have no way to tell when it was actually completely dissolved (and thus no way to calculate our way backwards to the initial state).

Exercise 7 – Boiling an egg

We have been given the heat convection equation,

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T, \tag{16}$$

with the boundary condition

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=a} = h\left(T - T_w\right). \tag{17}$$

We let

- $T = T_0 + (T_1 - T_0)\, T'$
- $T_w = T_0 + (T_1 - T_0)\, T'_w$
- $t = \dfrac{a^2}{\kappa} t'$ (where $\kappa = \dfrac{k}{\rho c}$)
- $x = a x'$
- $y = a y'$.

If we insert these scalings into (16) we get

$$\frac{\partial T'}{\partial t'} = \nabla^2 T'. \tag{18}$$

The boundary condition in (17) reads

$$\left. \frac{\partial T'}{\partial r'} \right|_{r'=1} = \mathrm{Bi}\left(T' - T'_w\right). \tag{19}$$

We see that the only dimensionless parameter in the problem is the Biot-number.

No, wait a moment! We still have no parameter including the time t_0. In agreement with our scalings we let $t_0 = (a^2/\kappa)\tau$, the water is heated during the dimensjonsløs tid τ. Notice that a^2/κ is a time constant that is characteristic for the egg: It is approximately the time required to complete heat the egg. This means that at dimensionless time $\tau \gg 1$ the egg is completely heated but if $\tau \ll 1$ is it only warm near the surface.