# Suggested solution Mathematical modelling
# Exercise 1, autumn 2005

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**Exercise 11 – Rowing**

The total volume occupied by the rowers is given by $NV$. Since

$$\begin{aligned}[NV] &= \left[\mathrm{m}^3\right]\\ [A] &= \left[\mathrm{m}^2\right],\end{aligned}$$

a dimensionless combination is given by $\pi_1 = A/(NV)^{2/3}$, and thus $A \propto (NV)^{2/3}$.

To show that the resistance force might be given by $F_d = \rho U^2 A$, we once again apply dimensional analysis.

$$[\rho U^2 A] = \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \mathrm{m}^2 = \frac{\mathrm{kg\,m}}{\mathrm{s}^2}\right] = [\mathrm{N}]\,.$$

This shows that this is indeed a plausible expression for the resistance force. (Another way to be sure of this is due to the fact that dimensional analysis only gives one dimensionless combination of $F_d$, $\rho$, $U$ and $A$).

To find the work required, we use the good old work = force × distance. This gives another factor $U$, and thus $W \propto \rho U^3 A$.

Finally we use energy balance to get the expression given in the book. This reads

$$\begin{aligned}NP &\propto \rho U^3 A\\ NP &\propto \rho U^3 (NV)^{2/3}\\ U^3 &\propto N^{1-2/3} P^1 \rho^{-1} V^{-2/3}\\ \Rightarrow U &\propto N^{1/9} P^{1/3} \rho^{-1/3} V^{-2/9}.\end{aligned}$$

With $P \propto m$ and $V \propto m$ we get $U \propto N^{1/9} \rho^{-1/3} m^{1/9}$, which implies that size is of slight advantage for a rower.

**Exercise 12 – Similarity solution to the heat equation**

We have been given the problem

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad x > 0, \quad t > 0, \qquad (1)$$
$$T(x,0) = 0, \quad T(0,t) = T_0 > 0.$$

The task at hand is to show that this problem has a similarity solution given by

$$T = T_0 F\left(\frac{x}{\sqrt{\kappa t}}\right). \qquad (2)$$

We start by inserting (2) into (1). After some simplifications this reads

$$-\frac{\xi}{2}F'(\xi) = F''(\xi)$$

where $\xi = x/\sqrt{\kappa t}$. We then let $y = F'$, insert this and solve the resulting ordinary differential equation. Its solution is given by

$$y = C_0 e^{\xi^2/4}.$$

We then integrate once more to find $F$. Here we have to apply a substitution given by $s' = \frac{\xi}{2}$ in order to get the right hand side to be of the form erf($\xi$). After some rearragements this reads

$$F = C_1 + C_0' \operatorname{erf}(\xi/2) \Rightarrow T = T_0(C_1 + C_0' \operatorname{erf}(\xi/2)).$$

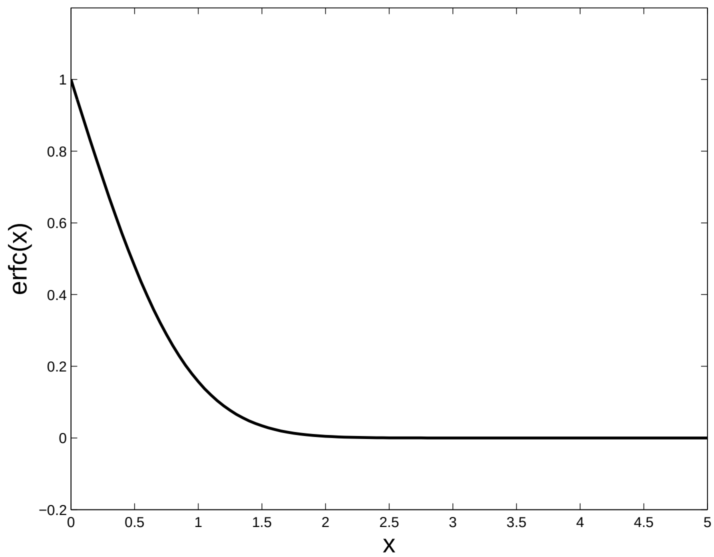
We then use the given boundary- and initial conditions to figure out the value of these constants. We get

$$T(0,t) = T_0(C_1 + C_0 \operatorname{erf}(0)) = T_0 C_1 = T_0 \Rightarrow C_1 = 1$$
$$T(x,0) = T_0(1 + C_0 \lim_{\xi\to\infty} \operatorname{erf}(\xi/2)) = T_0(1 + C_0) = 0 \Rightarrow C_0 = -1.$$

In other words,

$$F = 1 - \operatorname{erf}(\xi/2), \quad T = T_0(1 - \operatorname{erf}(\xi/2)) = T_0 \operatorname{erfc}(\xi/2).$$

This is called a similarity solution to the heat equation since both functions only depend on the parameter $\xi$.




Figur 1: Plot of the function erfc($\xi$) against $\xi$.

The function decreases rapidly towards 0 until $\xi \approx 1$. This fits nicely with the discussion given at the end of Section 2.2 in the book. If $\xi = 1$, we have $x = \sqrt{\kappa t}$. This means that the temperature change has propagated this far into the material and we expect only minor changes.

**Exerice 13 – Firewalking**

It is a known fact that the heat flux is given by $Q = k\nabla T$. With $T = T_0 \operatorname{erfc}\big(x/(2\sqrt{\kappa t})\big)$ the heat flux in $x = 0$ is given by

$$Q = -\frac{kT_0}{\sqrt{\pi\kappa t}} = -T_0\sqrt{\frac{\rho c k}{\pi t}}$$

(recall that $\operatorname{erfc}'(0) = -2/\sqrt{\pi}$ and $\kappa = k/(\rho c)$). This does not coincide with the solution given in the book. There might be a typographical error. Or perhaps the author thought of the heat flux integrated over one time interval $t$. If that is the case, we get (omitting the sign)

$$q = \int_0^t T_0\sqrt{\frac{\rho c k}{\pi\tau}}\,\mathrm{d}\tau = 2T_0\sqrt{\frac{\rho c k t}{\pi}}.$$

This fits with the solution given in the book, so this is probably what was intended.

If we insert $T_0 = 500\,\mathrm{K}$, $t = 0.5\,\mathrm{s}$ and the given values for wodden coal we end up with $q \approx 80\,\mathrm{kJ/m^2}$.

Let the penetration depth be given by $\delta$ ($\delta = 1\,\mathrm{mm}$). A coarse approximation for the temperature change in the foot is given by

$$\Delta T = \frac{q}{\rho c\delta} \approx 19\,\mathrm{K}.$$

We can do better than this. We can connect two similarity solutions, using the results found in Exercise 12. But we need to generalize the solution to have an initial condition $T(x,0) = T_1$ (with, as before, the boundary condition $T(0,t) = T_0$). This gives

$$T = T_1 + (T_0 - T_1)\operatorname{erfc}\Big(\frac{x}{2\sqrt{\kappa t}}\Big),$$

resulting in a heat flux

$$|Q| = |T_1 - T_0|\sqrt{\frac{\rho c k}{\pi t}}.$$

We now want to connect two materials with different properties, one with initial temperature given by $T(0) = T_1$, and one with intial temperature given by $T(0) = T_2$. After (an unknown) time the heat flux between the two materials will be equal and we will have a constant temperature $T_0$ at the boundary between them. This occours when

$$(T_2 - T_0)\sqrt{\rho_2 c_2 k_2} = (T_0 - T_1)\sqrt{\rho_1 c_1 k_1}.$$

The temperature at the intersection between the materials is given by

$$T_0 = \frac{T_1\sqrt{\rho_1 c_1 k_1} + T_2\sqrt{\rho_2 c_2 k_2}}{\sqrt{\rho_1 c_1 k_1} + \sqrt{\rho_2 c_2 k_2}}.$$

We need a value for $k_{\text{foot}}$, we here use the value for water, $k = 0.591$ (in SI units). Inserting this value (among others) we get

$$\Big(\sqrt{\rho c k}\Big)_{\text{tre}} = 219, \quad \Big(\sqrt{\rho c k}\Big)_{\text{fot}} = 1575, \quad \Big(\sqrt{\rho c k}\Big)_{\text{stål}} = 14421.$$

This shows that the surface temperature will be

$$T_0 \approx 0.88\,T_{\text{fot}} + 0.12\,T_{\text{tre}},$$

quite close to the foot's initial temperature.

But if we walk on steel, the surface temperature will be

$$T_0 \approx 0.1\,T_{\text{fot}} + 0.9\,T_{\text{stål}},$$

close to the steel's initial temperature. Bad for your feet!