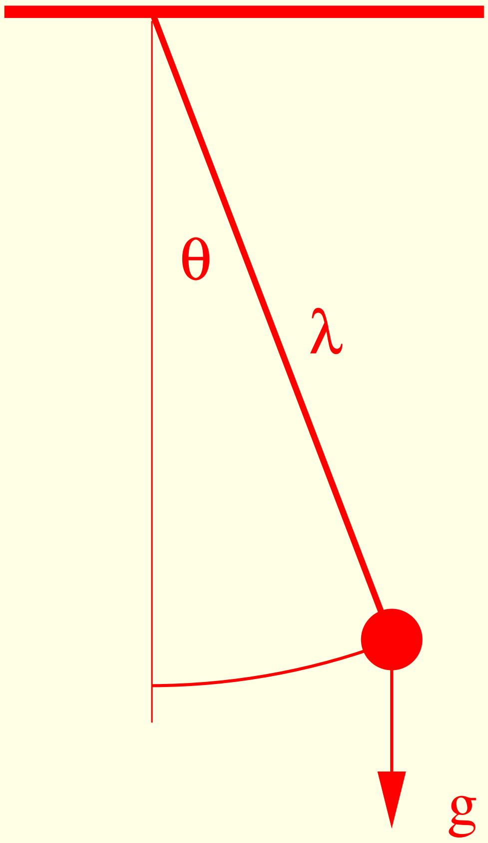
Buckingham's magical pi-theorem

Or: How to get a free lunch

Harald Hanche-Olsen

August 22, 2005

A simple example



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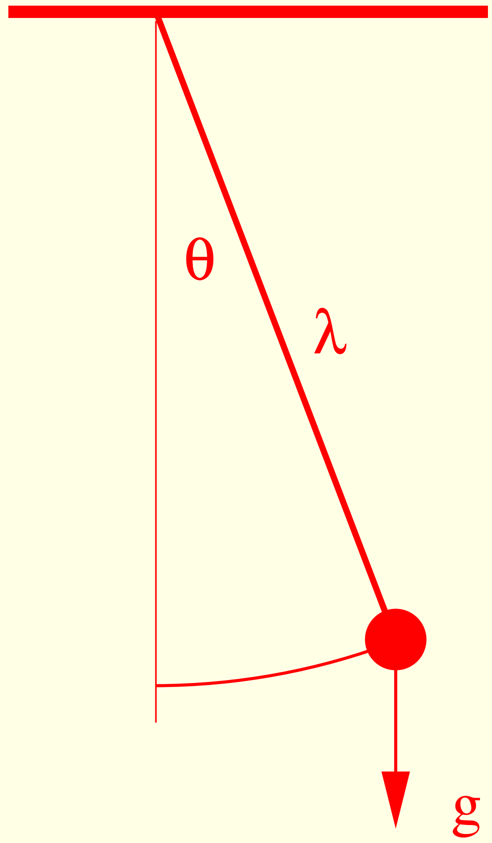

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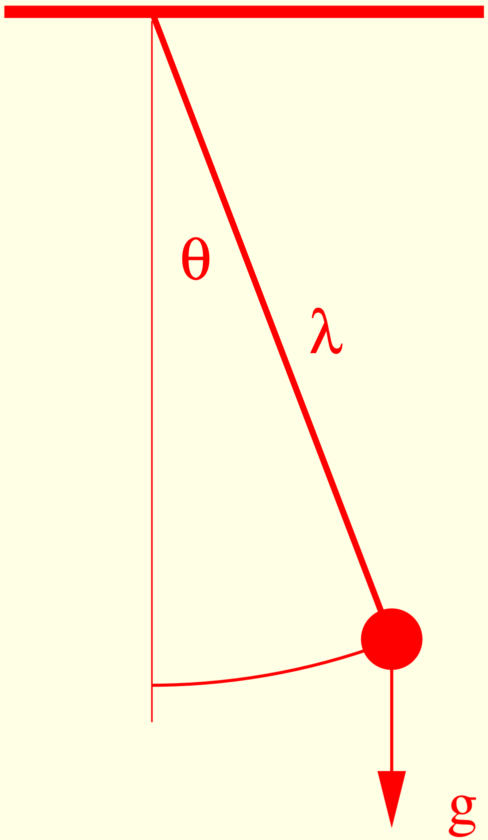

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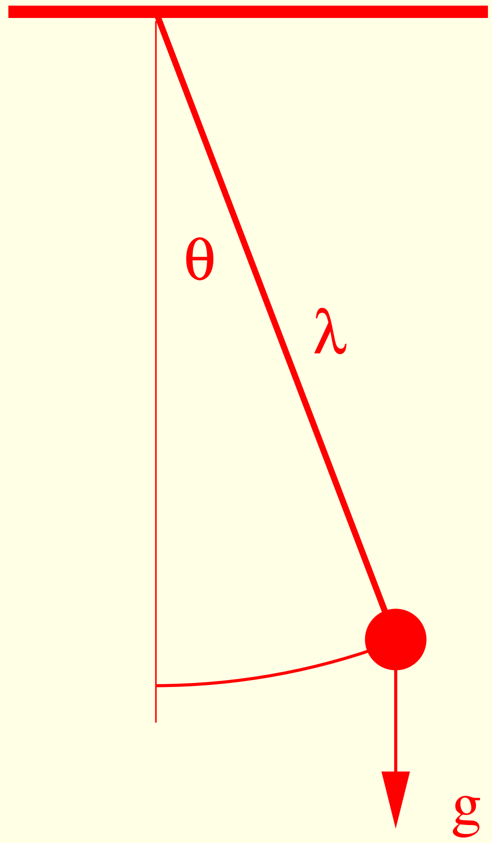

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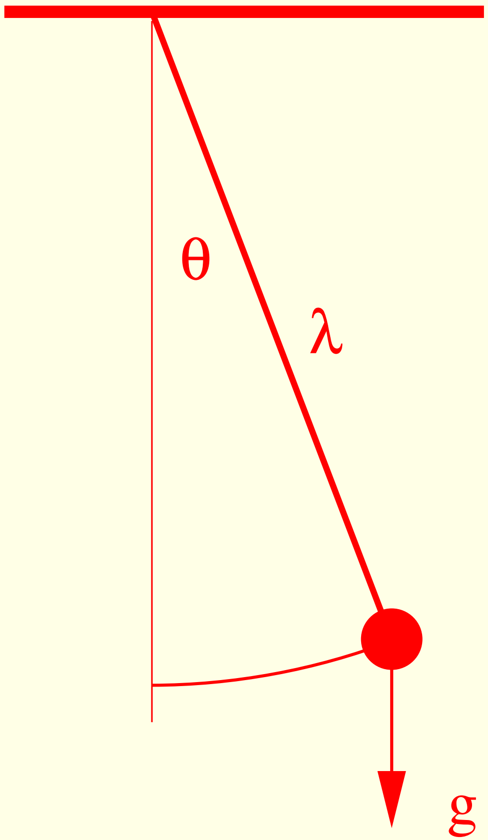

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Buckingham's pi theorem formalizes this procedure.

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and get the dimensionless form of the equation:

$$\ddot{\theta} + \sin\theta = 0$$

Edgar Buckingham (1867–1940)

Educated at Harvard and Leipzig, worked at the (US) National Bureau of Standards 1905–1937. (Soil physics, gas properties, acoustics, fluid mechanics, blackbody radiation.)

On Physically Similar Systems: Illustrations of the Use of Dimensional Equations. *Physical Review* **4**, 345–376 (1914).

The original paper

Vol. IV.] No. 4. ON PHYSICALLY SIMILAR SYSTEMS. 345

ON PHYSICALLY SIMILAR SYSTEMS; ILLUSTRATIONS OF THE USE OF DIMENSIONAL EQUATIONS.

By E. Buckingham.

1. *The Most General Form of Physical Equations.*—Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of n different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain n symbols $Q_1 \cdots Q_n$, one for each kind of quantity, and also, in general, a number of ratios r', r'', etc., so that it may be written

$$f(Q_1, Q_2, \cdots Q_n, r', r'', \cdots) = 0. \qquad (1)$$

Let us suppose, for the present only, that the ratios r do not vary during the phenomenon described by the equation: for example, if the equation describes a property of a material system and involves lengths, the system shall remain geometrically similar to itself during any changes of size which may occur. Under this condition equation (1) reduces to

$$F(Q_1, Q_2, \cdots Q_n) = 0. \qquad (2)$$

If none of the quantities involved in the relation has been overlooked, the equation will give a complete description of the relation subsisting among the quantities represented in it, and will be a complete equation.

The Framework

Physical quantities: $W_1, W_2, \ldots, W_n$

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What are the units of $\mathbf{W}^{\mathbf{x}}$?

How units combine

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Introduce the dimension vectors: $\{W_j\} = (a_{1j}, \ldots, a_{mj})^{\mathsf{T}}$ which form the columns of the dimension matrix $\mathbf{A}$. Formally:

$$[\mathbf{W}^{\mathbf{x}}] = \mathsf{L}^{\mathbf{A}\mathbf{x}}$$

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i.e., by linear algebra!

The combination $[\mathbf{W}^{\mathbf{x}}]$ is dimensionless iff $\mathbf{Ax} = 0$.

Dimensionless combinations

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Combinations $\mathbf{W}^{\mathbf{z}_\nu}$ are called independent if the vectors $\mathbf{z}_\nu$ are linearly independent.

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We can create a
maximal independent set of dimensionless combinations

$$\Pi_\nu = \mathbf{W}^{\mathbf{z}_\nu}, \quad \nu = 1, \ldots, k$$

by letting $\mathbf{z}_1, \ldots, \mathbf{z}_k$ be a basis for $\ker \mathbf{A}$.

New variables

By expanding to a basis for $\mathbb{R}^n$, we get an independent set of combinations:

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Key observation:
No nontrivial combination of $X_1, \ldots, X_{n-k}$ is dimensionless.
I.e., the vectors $\{X_1\}, \ldots, \{X_{n-k}\}$ are linearly independent.

Some physics

A general physical law:

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Since F is a result of computing with $(W_1, \dots, W_n)$, the units of F must be the units of a combination of $(W_1, \dots, W_n)$.

Thus we may assume WOLOG that F is dimensionless.

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The invariance under change of units thus means

$$F(W_1, \ldots, W_n) = F(e^{\mathbf{c}\{W_1\}} W_1, \ldots, e^{\mathbf{c}\{W_n\}} W_n) \qquad (\mathbf{c} \in \mathbb{R}^m)$$

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i.e., the equation is invariant under the action of an m-parameter group of symmetries.

Buckingham's theorem

Any dimensionally correct relationship involving physical quantities can be expressed in terms of a maximal set of dimensionless combinations of the given quantities:

$$\Phi(\Pi_1, \ldots, \Pi_k) = 0.$$

The proof

Rewrite the physical law in terms of these variables:

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Linear algebra tells us that the vectors

$$(\mathbf{c}\{X_1\}, \ldots, \mathbf{c}\{X_{n-k}\}), \qquad \mathbf{c} \in \mathbb{R}^m$$

fill all of $\mathbb{R}^{n-k}$, and it follows that
Φ depends only on the dimensionless combinations $(\Pi_1, \ldots, \Pi_k)$.

Proof detail

Recall: the vectors $\{X_1\}, \ldots, \{X_{n-k}\}$ are linearly independent.

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We make an $m \times (n-k)$ matrix

$$\mathbf{B} = \big(\{X_1\}, \ldots, \{X_{n-k}\}\big)$$

and note that when $\mathbf{c} \in \mathbb{R}^m$ (a row vector) then

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Since $\mathbf{B}$ has independent columns, it has rank $n-k$, and so its left image is all of $\mathbb{R}^{n-k}$ as claimed.

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Thus

$$v \propto \sqrt{g\lambda}.$$

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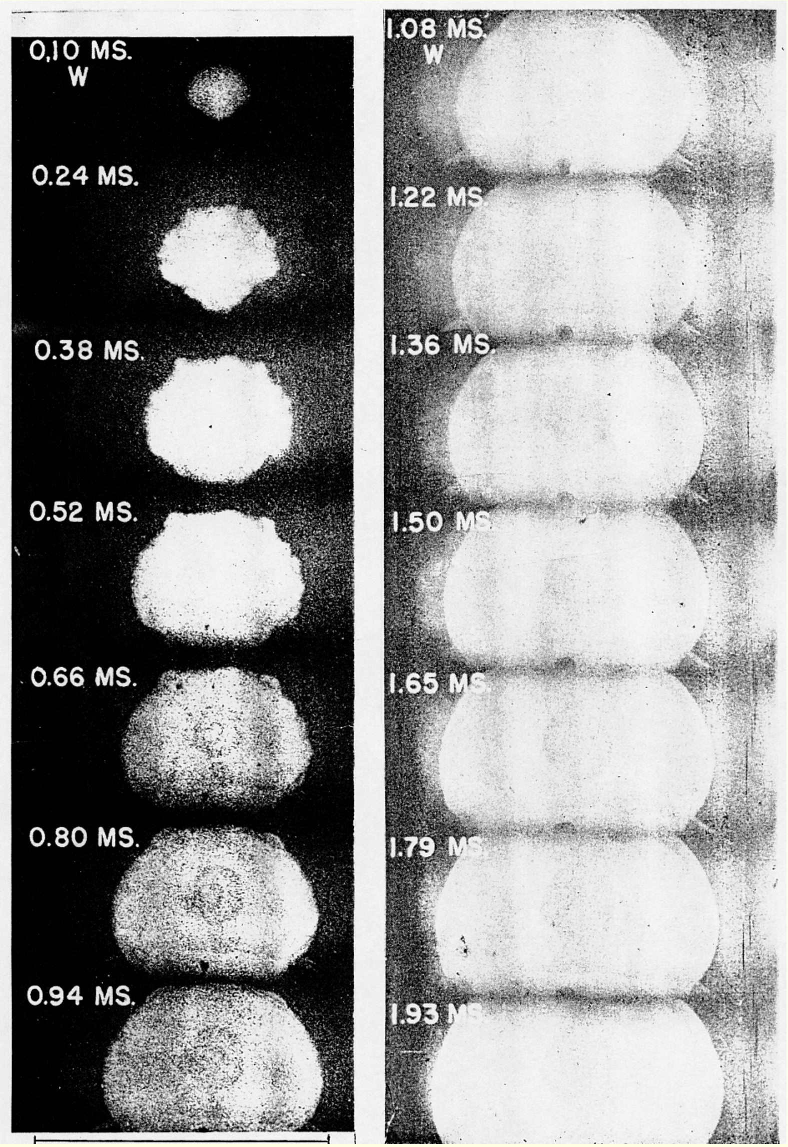
Just one dimensionless combination:

$$\Pi = r^5 t^{-2} \rho_0 E^{-1}$$

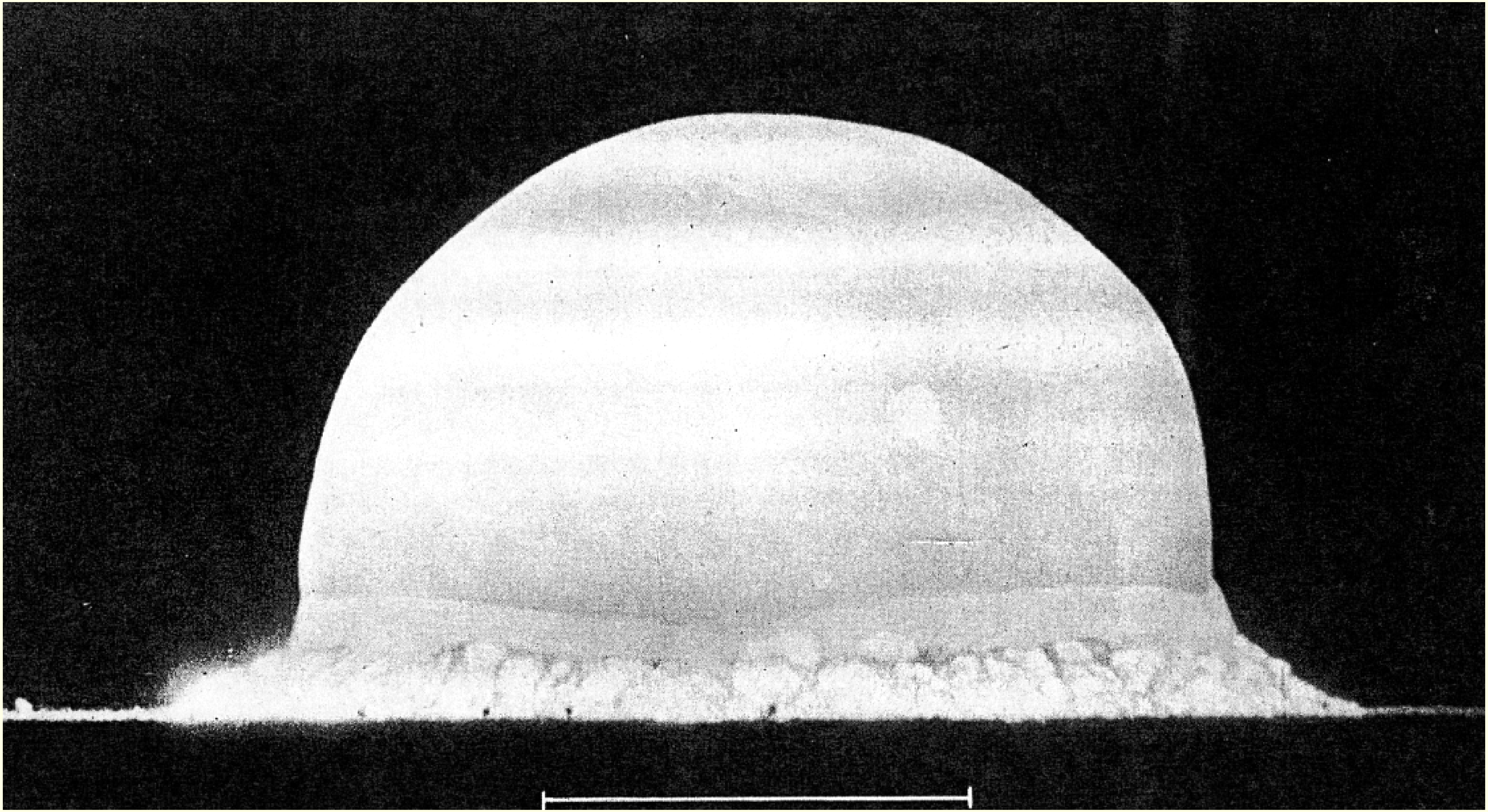
leading to

$$r \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}.$$

Nuclear explosion: First 2 ms

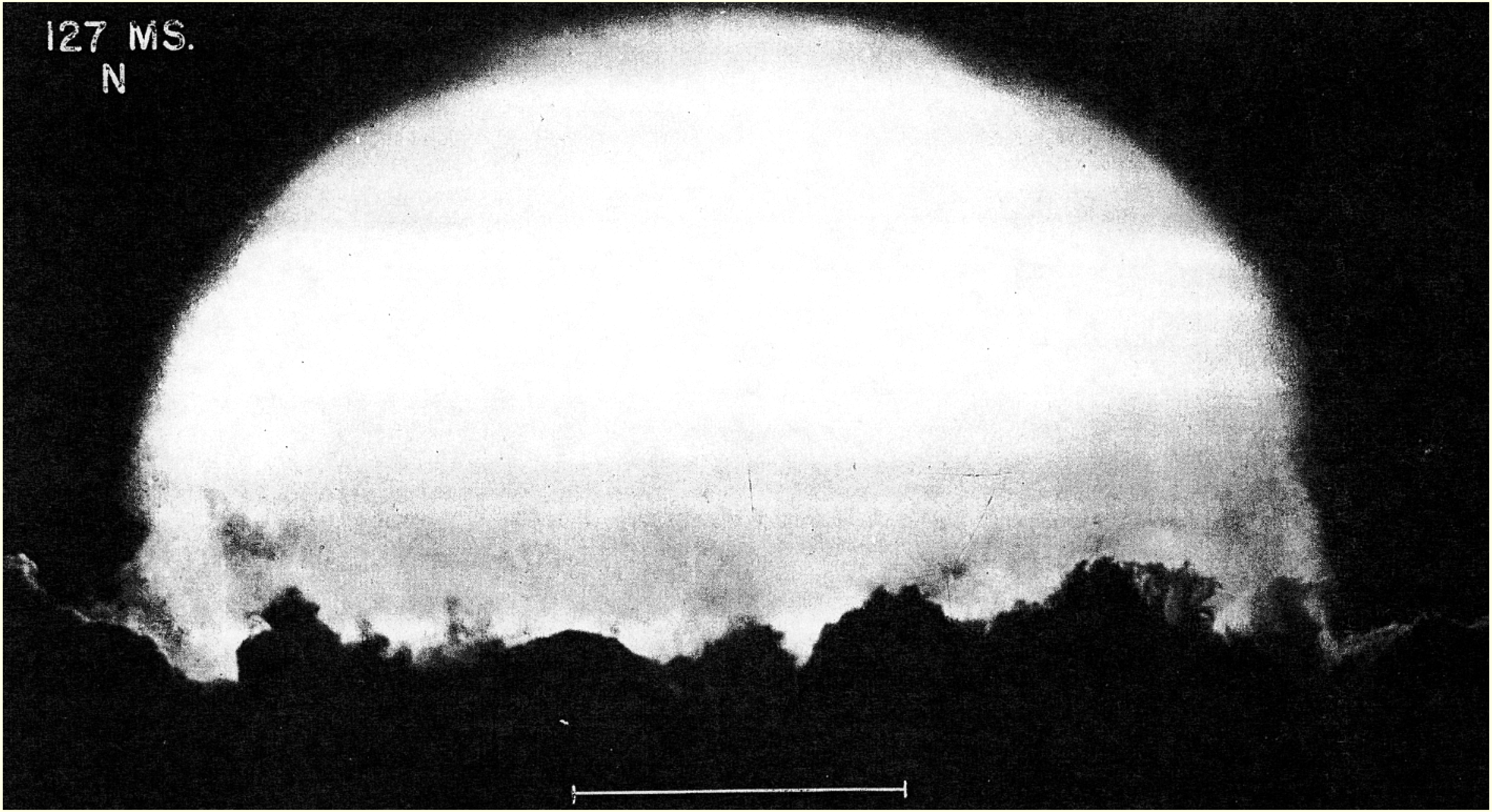


Nuclear explosion at 15 ms



The bar is 100 m long

Nuclear explosion at 127 ms



Nuclear yield

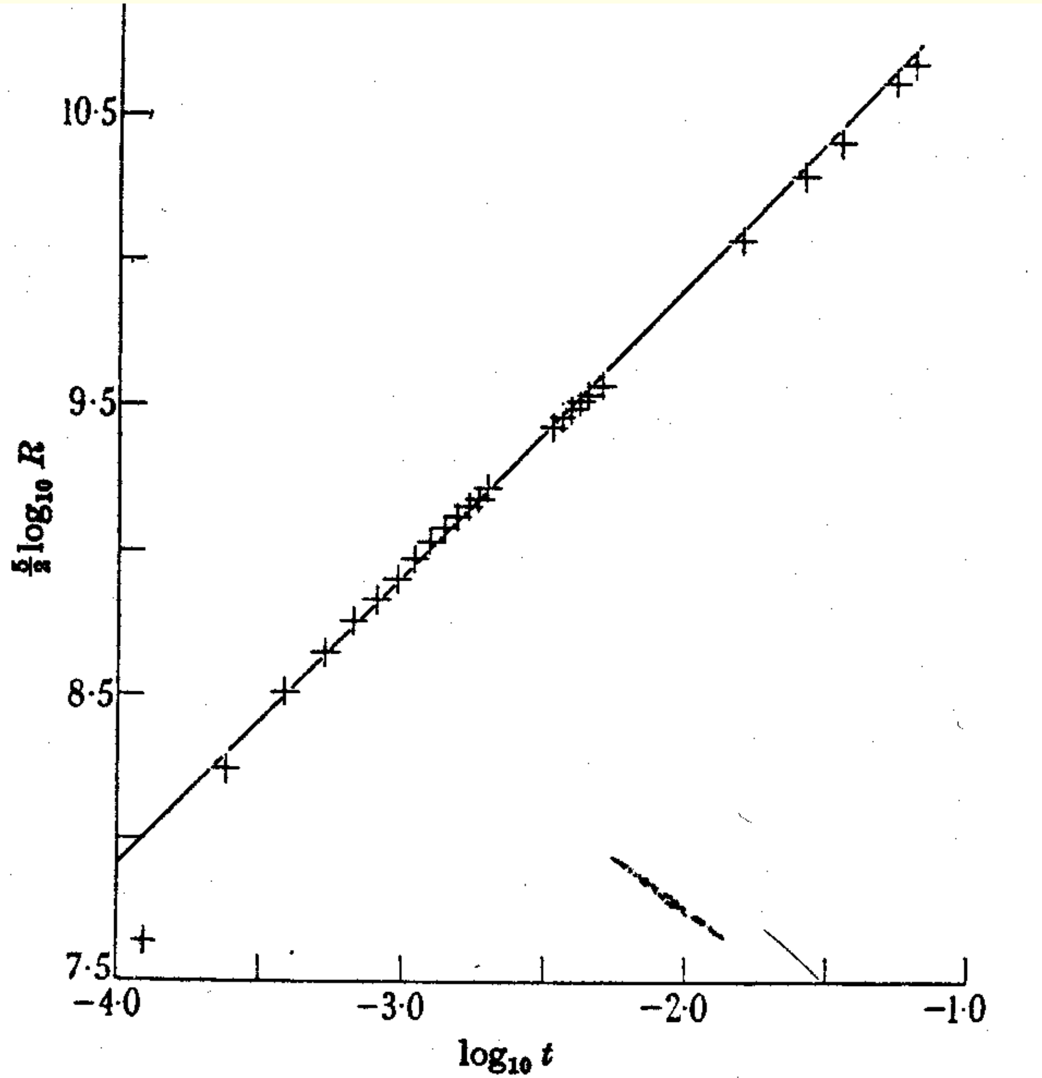


FIGURE 1. Logarithmic plot showing that $R^{\frac{5}{2}}$ is proportional to t.

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Pressure drop per unit length dP/dx as a function of density ρ, viscosity μ, average flow velocity U, pipe diameter D:

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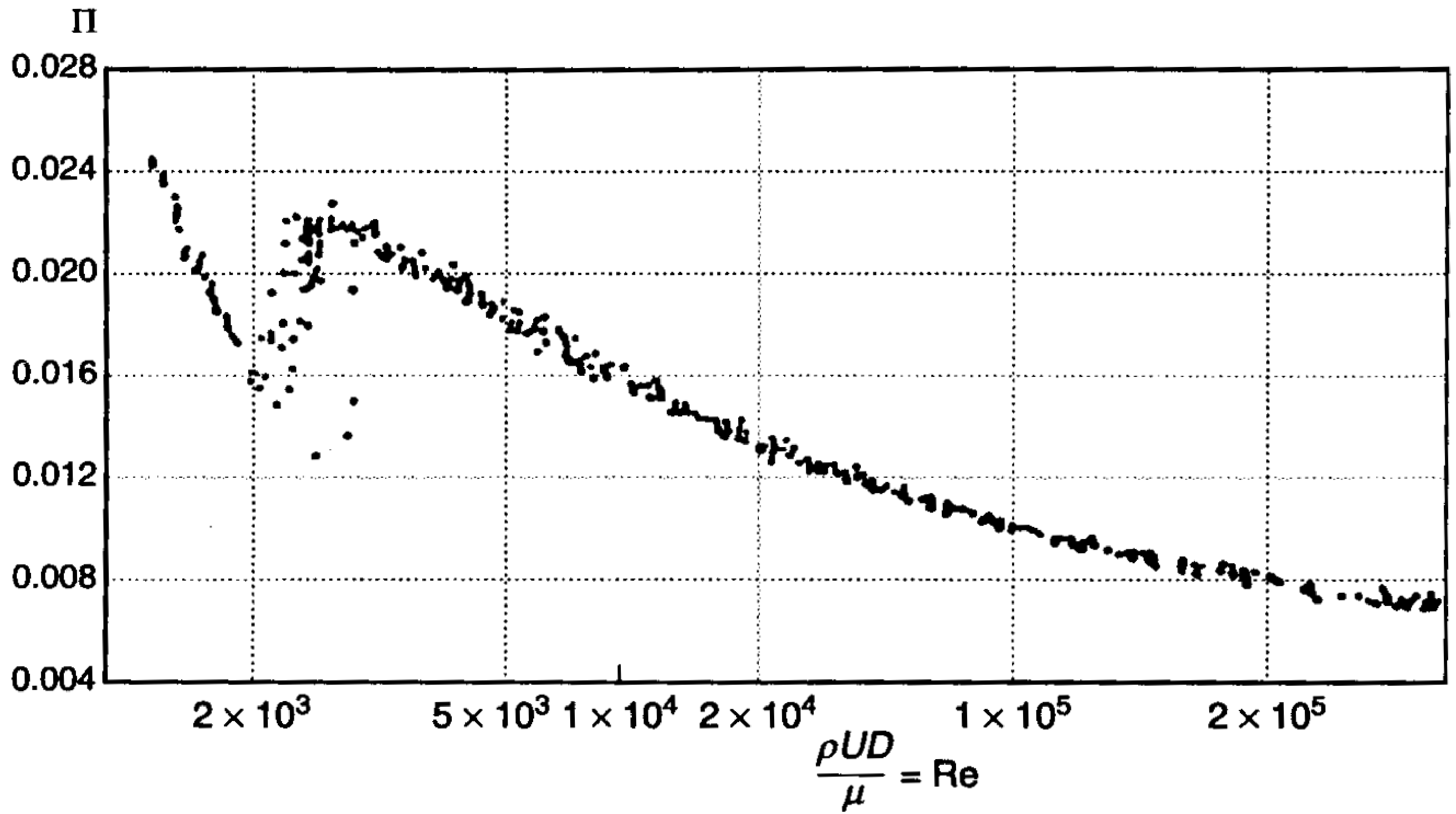
So we expect the dimensionless pressure Π to be a universal function of the Reynolds number Re:

$$\Pi = f(\mathrm{Re})$$

Experiments bear this out.

The Moody diagram



Symmetry and heat conduction

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By selecting the length scale L and time scale T so that $X^2/T = k/(\rho c)$ we get the dimensionless form of the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

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This leaves a degree of freedom still: Rescaling $x = \alpha x^*$, $t = \alpha^2 t^*$ yields the equation invariant.

In addition, we have the obvious scale invariance on u, since the equation is linear.

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If u solves this problem, then so does $\alpha u(\alpha x, \alpha^2 t)$.

Similarity solution

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$$u(x, t) = \alpha u(\alpha x, \alpha^2 t)$$

for all x, t, α.

Similarity solution

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$$u(x,t) = \alpha u(\alpha x, \alpha^2 t)$$

for all x, t, α.

With $\alpha = 1/\sqrt{t}$ this leads to

$$u(x,t) = \frac{1}{\sqrt{t}} v\Big(\frac{x^2}{t}\Big)$$

and a corresponding ODE for v. The final solution is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\Big(-\frac{x^2}{4t}\Big).$$

Sophus Lie (1842–1899)

Lie Symmetries

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If the surface $F = 0$ is invariant: A symmetry group of the ODE.

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But not all symmetries are due to rescaling.

Emmy Noether (1882–1935)

Variational problems

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Newton's law for a free particle in a potential field V.

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Noether's theorem establishes a one-to-one correspondence betwwen variational symmetries and integrating factors, and hence invariants, of the Euler–Lagrange equations.

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Rotational symmetry implies the conservation of angular momentum.

And time invariance implies the conservation of energy