# Hydraulic jump

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This little note is a supplement to the next-to-last part of the course notes, pp. 43–48. I am not going to rederive the equations here. (But I will remark that life becomes a little bit simpler if you chose $\theta_0 = \pi/2$ in the control volume for the impulse balance.

The mass balance and impulse balance become equations (190) and (196) in the compendium (in compact but hopefully unambiguous notation):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2} hr\,\mathrm{d}r + \Big[rhv\Big]_{r_1}^{r_2} = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2} rhv\,\mathrm{d}r + \Big[rhv^2 + \tfrac{1}{2}grh^2\Big]_{r_1}^{r_2} = \int_{r_1}^{r_2}\Big(\frac{1}{2}gh^2 - C_f v^2 r\Big)\mathrm{d}r.$$

Assuming stationary flow, we throw away the first term in each equation (with the time derivative). Further ignoring the friction term (i.e., setting $C_f = 0$) and assuming a smooth solution, we end up with the two equations

$$(rhv)' = 0, \tag{1}$$
$$(rhv^2 + \tfrac{1}{2}grh^2)' = \tfrac{1}{2}gh^2 \tag{2}$$

where the prime means differentiation with respect to $r$.

With all these assumptions, Bernoulli's law really should be built into these equations. And it is!

First, note that the first term in (2) is $(rhv^2)' = (rhv)'v + rhvv' = rhvv'$ by the product rule and (1).

Second, note that that the second term is $(\frac{1}{2}grh^2)' = \frac{1}{2}gh^2 + grhh'$ which partially cancels the right hand side of (2), and we are left with $rhvv' + grhh' = 0$. After dividing by $rh$, we are left with

$$(\tfrac{1}{2}v^2 + gh)' = 0 \tag{3}$$

which really is Bernoulli's law applied to a streamline either following the surface or the bottom of the flow.

The two equations (1) and (3) can be integrated to yield

$$rhv = M, \quad v^2 + 2gh = E \tag{4}$$

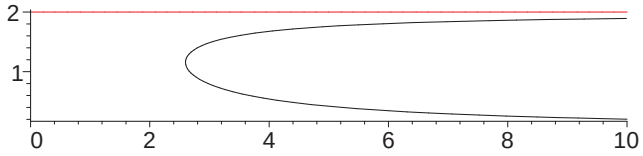




for constants $M$ (the total volumetric flow, divided by $2\pi$) and $E$ (twice the energy per unit mass of the flow). From the first equation we get $h = M/(rv)$ which we substitute into the second, getting $v^2 + 2gM/(rv) = E$. Perhaps more usefully, we write this as

$$\frac{2gM}{r} = (E - v^2)v, \tag{5}$$

and plot the result as follows, with $r$ along the horisontal axis and $v$ on the vertical axis. (I have arbitrarily plotted the graph with $E = 2$ and $2gM = 1$. Obviously, the general graph is a rescaled version of this one.)




We note that there are two solutions for a given (big enough) $r$: We might call the upper one the *fast* solution and the lower one the *slow* solution. It seems reasonable to expect that fast solution to be appropriate *inside*, and the slow one *outside* the hydraulic jump.

Differentiation the righthand side of (5) wrt $v$ we see that the turning point is at $v = (\frac{1}{3}E)^{1/2}$.

It is useful to express this in terms of the *Froude number*

$$\mathrm{Fr} = \frac{v}{(gh)^{1/2}}.$$

Recall that $(gh)^{1/2}$ is the *wave speed* for shallow water, so that $\mathrm{Fr} > 1$ means the water flow is faster than the wave speed. Using (4) first, and then (5) to eliminate $r$ we get

$$\mathrm{Fr}^2 = \frac{v^2}{gh} = \frac{rv^3}{gM} = \frac{2v^3}{(E - v^2)v} = \frac{2v^2}{E - v^2}$$

so that

$$\mathrm{Fr} > 1 \Leftrightarrow 2v^2 > E - v^2 \Leftrightarrow v > (\tfrac{1}{3}E)^{1/2},$$

which shows that $\mathrm{Fr} > 1$ on the upper branch of the curve and $\mathrm{Fr} < 1$ on the lower branch.

## The jump

We return now to the original equations on integral form. Again, looking for a stationary jump at $r$ we drop the time differentiated terms and let $r_1 \to r$ from below and $r_2 \to r$ from the right. The integral vansishes in the limit, and we end up with the two jump conditions (after dividing by the common factor $r$)

$$h_+ v_+ = h_- v_-, \quad h_+ v_+^2 + \tfrac{1}{2} g h_+^2 = h_- v_-^2 + \tfrac{1}{2} g h_-^2 \tag{6}$$

One way to solve this is the following trick: Note that

$$\frac{h v^2 + \frac{1}{2} g h^2}{(h v)^{4/3}} = \frac{v^{2/3}}{h^{1/3}} + \tfrac{1}{2} g \frac{h^{2/3}}{v^{4/3}} = g^{1/3} (\mathrm{Fr}^{2/3} + \tfrac{1}{2} \mathrm{Fr}^{-4/3})$$

(where we used $v^2/h = g\mathrm{Fr}^2$). So, in the second equation of (6) we divide the two sides by the 4/3rd power of the respective sides of the first equation, which yields

$$\mathrm{Fr}_+^{2/3} + \tfrac{1}{2} \mathrm{Fr}_+^{-4/3} = \mathrm{Fr}_-^{2/3} + \tfrac{1}{2} \mathrm{Fr}_-^{-4/3}.$$

However $\mathrm{Fr}^{2/3} + \frac{1}{2}\mathrm{Fr}^{-4/3}$ decreases from $\infty$ to $\frac{3}{2}$ for $\mathrm{Fr} \in (0,1]$, and increases again to $\infty$ for $\mathrm{Fr} \in [1,\infty)$. Thus to each $\mathrm{Fr}_- \in (1,\infty)$ there corresponds a unique solution $\mathrm{Fr}_+ \in (0,1)$, In other words, there is a possible jump from a fast flow to a slow one. (The equations also admit a jump from a slow flow to a fast one, but we don't believe in the physical possibility of such a flow.)

**Energy loss.** The quantity $e = \frac{1}{2} v^2 + g h$ is the total specific energy of a fluid particle, is preserved along a streamline according to Bernoulli's law. However, it will *not* be preserved across the hydraulic jump. The reason is that the region of the jump is a very turbulent region in which liquids with very different speeds collide, so energy is lost there.

To quantify this we use a trick similar to the one above, noting that

$$\frac{\frac{1}{2} v^2 + g h}{(v h)^{2/3}} = g^{2/3} (\tfrac{1}{2} \mathrm{Fr}^{4/3} + \mathrm{Fr}^{-2/3}).$$

The right hand side will change across the jump, and since the denominator on the left hand side does not change, the numerator must.

Experimenting a bit with Maple leads me to believe that

$$(\tfrac{1}{2} \mathrm{Fr}^{4/3} + \mathrm{Fr}^{-2/3}) - (\mathrm{Fr}^{2/3} + \tfrac{1}{2} \mathrm{Fr}^{-4/3})$$





is a strictly increasing function of Fr. Therefore, since $\mathrm{Fr}^{2/3} + \frac{1}{2}\mathrm{Fr}^{-4/3}$ is preserved across the jump, the energy will decrease (or increase) across the jump if and only if Fr decreases (or increases).

Indeed, with the shorthand notation $x = \mathrm{Fr}^2/3$ we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\big((\tfrac{1}{2}x^2 + x^{-1}) - (x + \tfrac{1}{2}x^{-2})\big) &= -x + x^{-2} + 1 - x^{-3} \\ &= x^{-3}(x^4 - x^3 - x + 1) = x^3(x-1)(x^3-1) > 0 \end{aligned}$$

when $x > 0$, $x \neq 1$.