Hydraulic jump

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This little note is a supplement to the next-to-last part of the course notes, pp. 43–48. I am not going to rederive the equations here. (But I will remark that life becomes a little bit simpler if you chose $\theta_0 = \pi/2$ in the control volume for the impulse balance.

The mass balance and impulse balance become equations (190) and (196) in the compendium (in compact but hopefully unambiguous notation):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2} hr\,\mathrm{d}r + \Big[rhv\Big]_{r_1}^{r_2} = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2} rhv\,\mathrm{d}r + \Big[rhv^2 + \tfrac{1}{2}grh^2\Big]_{r_1}^{r_2} = \int_{r_1}^{r_2}\Big(\frac{1}{2}gh^2 - C_f v^2 r\Big)\,\mathrm{d}r.$$

Assuming stationary flow, we throw away the first term in each equation (with the time derivative). Further ignoring the friction term (i.e., setting $C_f = 0$) and assuming a smooth solution, we end up with the two equations

$$(rhv)' = 0, \tag{1}$$
$$(rhv^2 + \tfrac{1}{2}grh^2)' = \tfrac{1}{2}gh^2 \tag{2}$$

where the prime means differentiation with respect to r.

With all these assumptions, Bernoulli's law really should be built into these equations. And it is!

First, note that the first term in (2) is $(rhv^2)' = (rhv)'v + rhvv' = rhvv'$ by the product rule and (1).

Second, note that that the second term is $(\frac{1}{2}grh^2)' = \frac{1}{2}gh^2 + grhh'$ which partially cancels the right hand side of (2), and we are left with $rhvv' + grhh' = 0$. After dividing by rh, we are left with

$$(\tfrac{1}{2}v^2 + gh)' = 0 \tag{3}$$

which really is Bernoulli's law applied to a streamline either following the surface or the bottom of the flow.

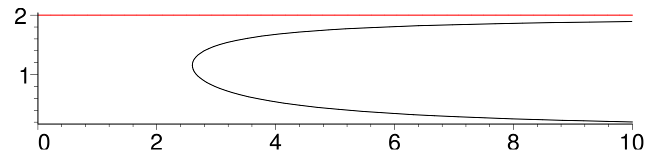
The two equations (1) and (3) can be integrated to yield

$$rhv = M, \quad v^2 + 2gh = E \tag{4}$$

for constants M (the total volumetric flow, divided by 2π) and E (twice the energy per unit mass of the flow). From the first equation we get $h = M/(rv)$ which we substitute into the second, getting $v^2 + 2gM/(rv) = E$. Perhaps more usefully, we write this as

$$\frac{2gM}{r} = (E - v^2)v, \tag{5}$$

and plot the result as follows, with r along the horisontal axis and v on the vertical axis. (I have arbitrarily plotted the graph with $E = 2$ and $2gM = 1$. Obviously, the general graph is a rescaled version of this one.)



We note that there are two solutions for a given (big enough) r: We might call the upper one the *fast* solution and the lower one the *slow* solution. It seems reasonable to expect that fast solution to be appropriate *inside*, and the slow one *outside* the hydraulic jump.

Differentiation the righthand side of (5) wrt v we see that the turning point is at $v = (\frac{1}{3}E)^{1/2}$.

It is useful to express this in terms of the *Froude number*

$$\mathrm{Fr} = \frac{v}{(gh)^{1/2}}.$$

Recall that $(gh)^{1/2}$ is the *wave speed* for shallow water, so that $\mathrm{Fr} > 1$ means the water flow is faster than the wave speed. Using (4) first, and then (5) to eliminate r we get

$$\mathrm{Fr}^2 = \frac{v^2}{gh} = \frac{rv^3}{gM} = \frac{2v^3}{(E - v^2)v} = \frac{2v^2}{E - v^2}$$

so that

$$\mathrm{Fr} > 1 \Leftrightarrow 2v^2 > E - v^2 \Leftrightarrow v > (\tfrac{1}{3}E)^{1/2},$$

which shows that $\mathrm{Fr} > 1$ on the upper branch of the curve and $\mathrm{Fr} < 1$ on the lower branch.

The jump

We return now to the original equations on integral form. Again, looking for a stationary jump at r we drop the time differentiated terms and let $r_1 \to r$ from below and $r_2 \to r$ from the right. The integral vansishes in the limit, and we end up with the two jump conditions (after dividing by the common factor r)

$$h_+ v_+ = h_- v_-, \quad h_+ v_+^2 + \tfrac{1}{2} g h_+^2 = h_- v_-^2 + \tfrac{1}{2} g h_-^2 \tag{6}$$

One way to solve this is the following trick: Note that

$$\frac{hv^2 + \frac{1}{2}gh^2}{(hv)^{4/3}} = \frac{v^{2/3}}{h^{1/3}} + \frac{1}{2}g\frac{h^{2/3}}{v^{4/3}} = g^{1/3}(\mathrm{Fr}^{2/3} + \tfrac{1}{2}\mathrm{Fr}^{-4/3})$$

(where we used $v^2/h = g\mathrm{Fr}^2$). So, in the second equation of (6) we divide the two sides by the 4/3rd power of the respective sides of the first equation, which yields

$$\mathrm{Fr}_+^{2/3} + \tfrac{1}{2}\mathrm{Fr}_+^{-4/3} = \mathrm{Fr}_-^{2/3} + \tfrac{1}{2}\mathrm{Fr}_-^{-4/3}.$$

However $\mathrm{Fr}^{2/3} + \frac{1}{2}\mathrm{Fr}^{-4/3}$ decreases from ∞ to $\frac{3}{2}$ for $\mathrm{Fr} \in (0,1]$, and increases again to ∞ for $\mathrm{Fr} \in [1,\infty)$. Thus to each $\mathrm{Fr}_- \in (1,\infty)$ there corresponds a unique solution $\mathrm{Fr}_+ \in (0,1)$, In other words, there is a possible jump from a fast flow to a slow one. (The equations also admit a jump from a slow flow to a fast one, but we don't believe in the physical possibility of such a flow.)

Energy loss. The quantity $e = \frac{1}{2}v^2 + gh$ is the total specific energy of a fluid particle, is preserved along a streamline according to Bernoulli's law. However, it will *not* be preserved across the hydraulic jump. The reason is that the region of the jump is a very turbulent region in which liquids with very different speeds collide, so energy is lost there.

To quantify this we use a trick similar to the one above, noting that

$$\frac{\frac{1}{2}v^2 + gh}{(vh)^{2/3}} = g^{2/3}(\tfrac{1}{2}\mathrm{Fr}^{4/3} + \mathrm{Fr}^{-2/3}).$$

The right hand side will change across the jump, and since the denominator on the left hand side does not change, the numerator must.

Experimenting a bit with Maple leads me to believe that

$$(\tfrac{1}{2}\mathrm{Fr}^{4/3} + \mathrm{Fr}^{-2/3}) - (\mathrm{Fr}^{2/3} + \tfrac{1}{2}\mathrm{Fr}^{-4/3})$$

is a strictly increasing function of Fr. Therefore, since $\mathrm{Fr}^{2/3} + \frac{1}{2}\mathrm{Fr}^{-4/3}$ is preserved across the jump, the energy will decrease (or increase) across the jump if and only if Fr decreases (or increases).

Indeed, with the shorthand notation $x = \mathrm{Fr}^{2/3}$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big((\tfrac{1}{2}x^2 + x^{-1}) - (x + \tfrac{1}{2}x^{-2})\big) &= -x + x^{-2} + 1 - x^{-3} \\ &= x^{-3}(x^4 - x^3 - x + 1) = x^3(x-1)(x^3-1) > 0\end{aligned}$$

when $x > 0$, $x \neq 1$.