TMA4195 Mathematical modelling 2004

Exercise set 9

Advice and suggestions: 2004–10–19

Exercise 1: (Exam December 1991.) A point particle is released from the point $(x^*, y^*) = (a, a^2/(2b))$ with initial velocity zero and thereafter slides without friction along the parabola $y^* = {x^*}^2/(2b)$, where the y axis is vertical.

(a) Write up the equation of motion for the particle, pick a suitable scaling and find the first two terms in a perturbation expansion ($x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$) valid for small values of $\varepsilon = a^2/b^2$.

(b) In the above perturbation expansion $x_1(t)$ contains a secular term (i.e., a term which does not remain bounded as $t \to \infty$). Rescale time using a factor $1 + \varepsilon c$ and pick c so that the secular term disappears. Compute the period of the oscillation with a relative error $O(\varepsilon^2)$.

Exercise 2: (Exam August 1999.) During an epidemic in a population (P), x^* individuals are susceptible to infection, y^* are infected and ill, while z^* have been vaccinated or are immune after having having been ill ($P = x^* + y^* + z^*$). An isolated group of ill people will get well according to the equation $dy^*/dt^* = -\lambda y^*$, where λ is a constant. A given fraction of those getting will become immune. A vaccination program is underway, giving the vaccine to susceptible persons at a constant rate.

(a) After scaling (with a time scale based on λ) the following model has been suggested:

$$\dot{x} = -\alpha x y + \varepsilon y - \kappa,$$
$$\dot{y} = \alpha x y - y.$$

Explain the basis for this model, and show that a suitable scaling gives this form. What is the equation for z?

(b) The course of the epidemic from given starting points will describe orbits in the xy plane. What area of the plane is physically acceptable? Find an expression for the orbits when nobody is being vaccinated.

(c) Assume that $0 < \kappa < \varepsilon < 1 < \alpha$. Make a rough sketch of the orbits in the phase plane.

(d) Under the assumptions of the previous point, the system as a stationary point (not physically admissible). Where is this point, and what is its type?