TMA4195 Mathematical modelling 2004

Exercise set 7

For the special activity week (week 41)

Exercise 1: (Exam, August 1996 – somewhat modified.) The capacity of a sprinter is, among other things, determined by the maximal force she is able to produce (in the horizontal direction). Write this force as MP, where M is the mass of the runner – so that P is the maximal acceleration, in some sense. In addition, "inner resistance" – friction in arms and legs, as well as the work involved in accelerating them back and forth – can be modelled as a counter force $MR(u^*)$, where the function $R(u^*)$ of the sprinter's speed u^* satisfies $R(0) = 0$. We assume furthermore that R is approximately linear. Dimensional reasoning leads to $R(u^*) = u^*/\tau$ where τ is a time constant. Measurements on different sprinters yield $P \approx 10\ \mathrm{m/s^2}$ and $\tau \approx 1$ s, and we will use these values below.

(a) Explain why a reasonable equation of motion for the runner in still air may be

$$M\frac{du^*}{dt^*} + M\frac{u^*}{\tau} + \tfrac{1}{2}\rho_{\mathrm{air}} C_D A {u^*}^2 = M p^*(t^*)$$

and explain in particular the origin of the expression for the air resistance. Use dimensional analysis where A is the cross section area of the runner (as seen from the front) and $C_D = C_D(\mathrm{Re})$. Here $\mathrm{Re} = Lu^*/\nu$, where L is a typical length and ν is the (kinematic) viscosity of air.

(b) Find and defend a suitable choice of scales for u^*, t^* and p^*. (Hint: What is the greatest acceleration? What is the greatest steady speed, assuming no air resistance?) With one reasonable choice of scales, the dimensionless form becomes

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \qquad \varepsilon = \tfrac{1}{2}\rho_{\mathrm{air}} C_D \frac{A}{M}\tau^2 P.$$

Is the air resistance very important? (Assume $C_d \approx 1$, $A \approx 0.45\,\mathrm{m^2}$, $\rho_{\mathrm{air}} \approx 1\,\mathrm{kg/m^3}$.)

(c) In a hundred meter dash we may assume that the runner uses maximal power all the way, i.e., $p^*(t^*) = P$. Determine the first two terms in a perturbation expansion for u in this case. How quickly after the start does a sprinter achieve a roughly constant speed?

It is well known that the wind will influence the times of a hundred meter dash. How is the friction term changed by a tail wind of speed W? Use $\delta = W/(\tau P)$ as a non-dimensional measure of the tail wind.

(d) Determine, up to order ε, the maximal speed a sprinter can maintain as a function of δ.

In one of the qualification heats during the Olympic games in 1988, Florence Griffith-Joyner beat the world record on the hundred meter dash (10.49 s). In the other runs and in the games themselves her times were around 10.7 s. The wind gauge registered no wind during the record run, but many people thought there was a considerable tail wind (estimated at 4 m/s) and that the gauge was out of order.

(e) Discuss this claim.

Exercise 2: (Lin & Segel page 300, problem 7.) For a slightly stiff string with fixed ends, the modes of vibration have a shape $y(x)$ given by the eigenvalue problem

$$\varepsilon y'''' - y'' = \lambda y, \qquad y(0) = y'(0) = y(1) = y'(1) = 0.$$

Here ε is a measure of the stiffness; $0 < \varepsilon \ll 1$. The eigenvalue λ is a dimensionless frequency of vibration. Find outer and inner approximations, assuming that λ is $O(1)$, and conclude that the stiffness has no effects on the eigenvalues to lowest order.