TMA4195 Mathematical modelling 2004

Exercise set 2

Advice and suggestions: 2004–08–31

This problem is a famous exam problem from 1989, when prof. Ola Bratteli (now at the University of Oslo) taught the class. The problem leads to relatively complicated expressions. It might be an advantage to use Maple to keep track. This text contains a few extra hints.

The method used in points c–d is called *Poincaré's method*. It is essentially the same method I used for the movement of a pendulum in the lecture on Thursday.

Exercise 1: (Exam 1989, somewhat rephrased).
The relativistic equation for a planet moving around the Sun is, in dimensionless form,

$$\frac{d^2 u}{d\theta^2} + u = 1 + \varepsilon u^2,$$

where $u = 1/r$, and r and θ are polar coordinates. The term εu^2 is a relativistic correction, $0 < \varepsilon \ll 1$.

(a) Show that the n-th order term in the perturbation expansion

$$u = u_0(\theta) + \varepsilon u_1(\theta) + \varepsilon^2 u_2(\theta) + \cdots$$

satisfies a differential equation on the form

$$\frac{d^2 u_n}{d\theta^2} + u_n = f_n(u_0, u_1, \ldots, u_{n-1}),$$

where f_0 is a constant and f_n is a function of n variables. Find f_n for all n.
(*Hint*: *You are not supposed to* solve *the equations, only to* find *them*!)

(b) Assume the initial condtions

$$u(0) = e + 1, \quad \frac{du}{d\theta}(0) = 0.$$

Here e is the eccentricity of the unperturbed orbit. With these starting conditions $\theta = 0$ corresponds to the *perihelion* of the orbit, i.e., the point of the orbit nearest the Sun ($u = 1/r$ is greatest). Find u_0 and u_1 using regular perturbation using the results from point a).
(*Hint*: *Remember that terms on the righthand side which also are solutions of the homogeneous equation lead to special solutions, such as* secular *terms of the form $\theta \cos\theta$ or $\theta \sin\theta$. These are unphysical because the have unlimited growth.*)

(c) To get rid of secular terms we will also develop the angle in a perturbation series. Introduce a new angular variable

$$\varphi = (1 + \varepsilon h)\theta$$

and let $v(\varphi) = u(\theta)$. Here h is a (for now, undetermined) constant. Transform the equation and the starting conditions so that the involve only v and φ. Show that there exists a value for h such that v_0 and v_1 in the new perturbation series

$$v = v_0(\varphi) + \varepsilon v_1(\varphi) + \varepsilon^2 v_2(\varphi) + \cdots$$

are periodic with period 2π with respect to φ, and find v_0 and v_1.

(d) Use the resultat from c) to show that the perihelion of the planet moves forward an angle $2\pi\varepsilon$ per orbit (to first order in ε).