Suggested solution for exercise Set A

a In component form the two given equations (1) and (2) become [1]

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{6}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu \nabla^{*2} u^* \tag{7}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -g - \frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \nu \nabla^{*2} v^* \tag{8}$$

With a scaling as indicated in the problem text we must have

$$\frac{U}{L} = \frac{V}{H} \tag{9}$$

for (6) to be correctly scaled, and (6) will then have the form

$$u_x + v_y = 0. \tag{10}$$

Moreover, turning (9) upside down we get two equal time scales, one "vertical" and one "horizontal" – so this will be a prime candidate for our time scale:

$$T = \frac{L}{U} = \frac{H}{V}$$

The pressure varies from 0 at the surface to $\approx \rho g H$ at the bottom, so

$$p^* = \rho g H p$$

(as in the problem text) seems reasonable. If we introduce this into (7) we get

$$\frac{U}{T} u_t + \frac{U^2}{L} u u_x + \frac{UV}{H} v u_y = -\frac{gH}{L} p_x + \nu U (L^{-2} u_{xx} + H^{-2} u_{yy})$$

where the three fractions on the left hand side are all equal, so a good scaling would make gH/L equal to these (we expect viscous forces to play a minor role, so the final part of the righthand side will be small). Thus we should have

$$U^2 = gH,$$

and a bit of computing yields

$$u_t + u u_x + v u_y = -p_x + \frac{1}{\varepsilon \mathrm{Re}} (\varepsilon^2 u_{xx} + u_{yy}), \tag{11}$$

while (8) becomes

$$\varepsilon^2 (v_t + u v_x + v v_y) = -1 - p_y + \frac{\varepsilon}{\mathrm{Re}} (\varepsilon^2 v_{xx} + v_{yy}). \tag{12}$$

[1] Equation numbers (1)–(5) in what follows refer to the equations of the problem text.

For tidal waves in the North Sea we find

$$\begin{aligned}
U &= \sqrt{gH} \approx 30\text{m/s},\\
L &= UT \approx 30\text{m/s}\cdot 6\cdot 3600\text{s} \approx 600\text{km},\\
\varepsilon &= H/L \approx 100\text{m}/600\text{km} \approx 2\cdot 10^{-4},\\
\text{Re} &= UH/\nu \approx 30\text{m/s}\cdot 100\text{m}/(10^{-6}m^2/s) = 3\cdot 10^9
\end{aligned}$$

so that $1/\varepsilon\text{Re} \approx 10^{-5}$ and it seems reasonable to expect that (10), (11) (12) can be simplified to the system (3).

b For a given fluid particle we find

$$\frac{d^2x}{dt^2} = \frac{du}{dt} = u_t + \frac{dx}{dt}u_x + \frac{dy}{dt}u_y = u_t + uu_x + vu_y = -p_x = -h_x$$

by using the middle equation in (3), and finally the relation $p = h(x,t) - y$ noted in the problem text. In particular, since h_x is independent of y, the uniqueness theory for second order differential equations shows that all fluid particle starting with the same x value (but different y) will continue to have the same x value for every future time, provided only that they all had the same value for $\dot{x} = u$ initially. This is precisely so when u is independent of y initially. So all the particles that shared the same x value initially will share the same x value in the future, and hence the same u value. So u is independent of y, as claimed.

This implies $u_y = 0$. So the leftmost part of equation (3) simplifies to the (4) as claimed (once again, we use $p_x = h_x$).

c A fluid particle on the surface satisfies $y = h(x,t)$. Differentiation of this relationship with respect to t yields

$$v = h_x u + h_t \quad \text{for } y = h(x,t)$$

Since $v = 0$ whenever $y = 0$ we find, using Green's formula:

$$\begin{aligned}
0 &= \iint_R (u_x + v_y)\,dx\,dy = \int_{\partial R} (-v\,dx + u\,dy)\\
&= \int_{x_1}^{x_2} \big((h_x u + h_t)\,dx - uh_x\,dx\big) + u(x_2,t)h(x_2,t) - u(x_1,t)h(x_1,t)\\
&= \int_{x_1}^{x_2} (h_t + (uh)_x)\,dx
\end{aligned}$$

where we have used that $u\,dy = uh_x\,dx$ along the curve $y = h(x,t)$. from this $h_t + (uh)_x = 0$ by the usual argument (i.e., using the Reymond–duBois lemma).

This is a standard conservation law in differential form, with conserved quantity h and flux uh. This makes sense, as the integral of h becomes an area (which becomes volume if we multiply by some arbitrary length along the suppressed space direction) and uh is the total transport of water past a given point (in m^2/s, which becomes m^3/s if we multiply by the same arbitrary length). So this equation expresses the conservation of volume, which is equivalent with conservation of mass so long as the density is considered constant.