Exercise Set 3

Exercise 1

A noticeable proportion of the heat input must penetrate to a depth of $L \approx 1\,\mathrm{cm}$ in $t \approx 30\,\mathrm{s}$ for the measurement to satisfy the given requirements. But how to estimate this depth? Thermal properties for water, fat, and protein have been given, and we must expect the properties for meat or fish to be some sort of average of these: Neither larger than the largest of the individual values, nor smaller than the smallest. The only dimensionless combination of the quantities L, t, ρ, c, k is $kt/(\rho c L^2)$ (or powers thereof), leading to the estimate

$$L \approx \sqrt{\frac{kt}{\rho c}}$$

in which we optimistically insert the largest available value for k (that for water) and the smallest one for c (protein) together with $t = 30\,\mathrm{s}$, or

$$L \approx \sqrt{\frac{0.56 \cdot 30}{1000 \cdot 1300}}\,\mathrm{m} \approx 3.6\,\mathrm{mm}$$

which is quite a bit too small, particularly because the heat must not only penetrate three times deeper, but the influence of the thermal properties at that depth must make it back to the surface where it can be measured. Thus we conclude that this device is likely to only at best yield properties of the upper millimeter or two of the sample, and so fails the requirements.

We should note that in the above discussion we replaced a constant arising from dimensional analysis by 1 without further comment. However, in the case of the heat equation and its fundamental solution (in one space dimension)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2}, \qquad \Phi(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{x^2}{4t}\Big)$$

this practice can be defended – or maybe we should use twice the value, as the exponent equals -1 just when $x = 2\sqrt{t}$ – but this does not change the conclusion very much.

One can also arrive more directly at these conclusions by recognizing that the non-scaled heat equation can be written in the form

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\Delta T$$

where the diffusivity (the fraction on the right hand side) has units $\mathrm{m\,s^{-1}}$, scaling the equation correspondingly, and using the above-mentioned properties of the solutions of the heat equation.

Exercise 2

We are looking for the zeros of

$$36x^3 + (162 + 4\varepsilon)x^2 - 24\varepsilon x - 9\varepsilon = 0. \tag{1}$$

We expand x as $x = x_0 + \varepsilon x_1 + o(\varepsilon)$. After plugging this expression into (1), we get, at the first order,

$$36x_0^3 + 162x_0^2 = 0.$$

Hence,

$$x_0 = -\frac{9}{2} \quad \text{or} \quad x_0 = 0 \quad \text{(double root)}.$$

At the second order, we get:

$$108x_0^2 x_1 + 4x_0^2 + 324x_0x_1 - 24x_0 - 9 = 0.$$

For $x_0 = -\frac{9}{2}$, it reduces to

$$729x_1 + 180 = 0$$

which gives us

$$x_1 = \frac{20}{81}.$$

For $x_0 = 0$, we get $-9 = 0$ which is impossible and x cannot be expanded as $x = x_0 + \varepsilon x_1$ near 0 (in other words, $x - x_0$ is not of order ε). We try an other power of ε: $x = \varepsilon^p x_1 + o(\varepsilon^p)$. We have

$$36\varepsilon^{3p}x_1^3 + 162\varepsilon^{2p}x_1^2 + 4\varepsilon^{2p+1}x_1^2 - 24\varepsilon^{p+1}x_1 - 9\varepsilon = 0.$$

$p = 1/2$ is the smallest strictly positive value which gives rise to more than one leading order term. Taking $p = 1/2$ leads us to

$$162x_1^2 = 9$$

and

$$x_1 = \pm\sqrt{\frac{9}{162}}.$$

Finally, first approximations of the roots are given by

$$x = -\frac{9}{2} + \frac{20}{81}\varepsilon$$

and

$$x = \pm\sqrt{\frac{9}{162}}\sqrt{\varepsilon}.$$

Exercise 3

We want to solve

$$\varepsilon y'' + (1+x^2)y' + y = 0 \tag{2}$$

with the boundary conditions

$$y(0) = 0,\ y(1) = 1.$$

The inner solution y_m is given by

$$(1+x^2)y_m' + y_m = 0$$

which can be integrated explicitly:

$$y_m = Ce^{-\arctan x}.$$

We choose the constant C so that the boundary condition on the right is fulfilled. We end up with

$$y_m = e^{\frac{\pi}{4} - \arctan x}$$

We want now to determine the scaling x_l for the outer solution on the left:

$$x_l = \frac{x}{\varepsilon^\alpha}.$$

We have

$$y_l(x_l) = y(x)$$

and

$$\begin{aligned} y'(x) &= y_l'(x_l)\frac{1}{\varepsilon^\alpha} \\ y''(x) &= y_l''(x_l)\frac{1}{\varepsilon^{2\alpha}}. \end{aligned}$$

Plugging that into equation (2), we get

$$\varepsilon^{1-2\alpha} y_l'' + (1+\varepsilon^{2\alpha} x_l^2)\varepsilon^{-\alpha} y_l' + y_l = 0. \tag{3}$$

The smallest α strictly bigger than zero which gives rise to more than one leading term is $\alpha = 1$ and then equation (3) yields

$$y_l'' + y_l' = 0.$$

The general solution of this equation is

$$y_l = Ae^{-x_l} + B.$$

The boundary condition $y(0) = 0$ implies that $B = -A$ and

$$y_l = A(1 - e^{-x_l}).$$

To determine A, we match the outer and the inner expansions:

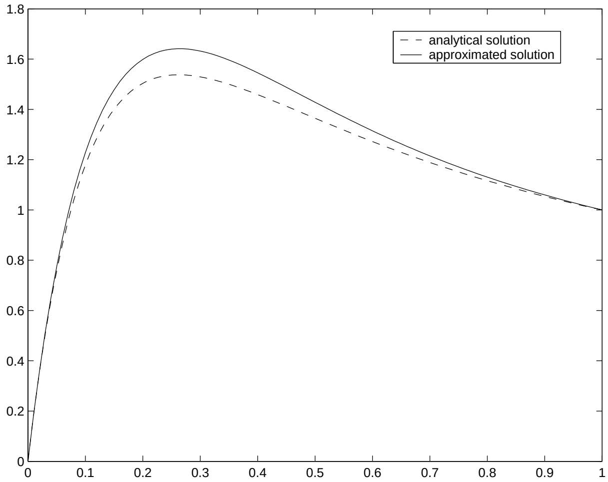
$$\lim_{x_l \to \infty} y_l(x_l) = \lim_{x \to 0} y_m(x).$$

Hence,

$$A = e^{\frac{\pi}{4}}.$$

The total expansion is

$$\begin{aligned} y &= y_l(x_l) + y_m(x) - \lim_{x_l \to \infty} y_l(x_l) \\ &= e^{\frac{\pi}{4}}(e^{-\arctan x} - e^{-\frac{x}{\varepsilon}}). \end{aligned}$$



Approximated and analytical solution for $\varepsilon = 0.1$