Exercise Set 8

a)

We can derive the governing equation by using lagrangian coordinates:

$$\tilde{x}(t,x)$$

$\tilde{x}(t,x)$ is the position at time t of the particule that was at x at time $t=0$.

The relation between Euler and Lagrangian coordinates is given by

$$v(t,\tilde{x}) = \frac{\partial \tilde{x}}{\partial \tilde{t}}$$

(For the time being, we drop the star in our notation)

The runners do not overtake each other. The number N of people running between a runner starting at $x = x_0$ and an other starting at $x = x_1$ remains the same.

$$N = \int_{\tilde{x}(t,x_0)}^{\tilde{x}(t,x_1)} \rho(t,u)\ du$$

N is constant in time. $\frac{\partial N}{\partial t} = 0$ implies

$$\int_{\tilde{x}(t,x_0)}^{\tilde{x}(t,x_1)} \rho_t(t,u)\ du + \tilde{x}_t(t,x_1)\rho(t,\tilde{x}(t,x_1)) - \tilde{x}_t(t,x_0)\rho(t,\tilde{x}(t,x_0)) = 0$$

Since

$$\tilde{x}_t(t,x) = v(t,\tilde{x}(t,x)),$$

this last equation can be rewritten

$$\int_{\tilde{x}(t,x_0)}^{\tilde{x}(t,x_1)} \rho_t(t,u)\ du + v(t,\tilde{x}(t,x_1))\rho(t,\tilde{x}(t,x_1)) - v(t,\tilde{x}(t,x_0))\rho(t,\tilde{x}(t,x_0)) = 0$$

hence,

$$\int_{\tilde{x}(t,x_0)}^{\tilde{x}(t,x_1)} \rho_t(t,u) + (v\rho)_x(t,u)\ du = 0$$

This equality holds for any x_0 and x_1 if and only if the integrand vanishes everywhere. We get

$$\rho_t + (v\rho)_x = 0$$

We now reintroduce the stars in our notation and rescale the problem.

$$\rho = \frac{\rho^*}{\rho^*_{max}},\ v = \frac{v^*}{v^*_{max}},\ x = \frac{x^*}{L}$$

are natural scaling. They induce the scaling

$$t = \frac{t^*}{v^*}$$

and we get

$$\rho_t + (v\rho)_x = 0 \tag{1}$$

With the scaled variables, the relation between speed and density becomes

$$v = 1 - \rho$$

Hence, from (1),

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

b)

The characteristics are the curves defined by

$$\frac{dx}{dt} = 1 - 2\rho \tag{2}$$

along which ρ is constant. They are straight lines.

we integrate (2) and get

$$x = (1 - 2\rho)t + \xi \tag{3}$$

where ξ is a constant. At $t = 0$, $x = \xi$ and ρ is then determined by

$$\rho = \rho_0 + \varepsilon \cos(\xi) \tag{4}$$

From (3) and (4), after eliminating ξ, we get an implicit solution for ρ

$$\rho = \rho_0 + \varepsilon \cos(x - (1 - 2\rho)t)$$

c)

Let c denote the characteristic speed

$$c = 1 - 2\rho$$

We have

$$x = c(\xi)t + \xi \tag{5}$$

A shock forms when

$$c'(\xi)t + 1 = 0 \tag{6}$$

because (5) is then no more invertible with respect to ξ ($\frac{\partial x}{\partial \xi}$ vanishes).

Since t can only be positive, equation (6) has a solution if and only if there exists a ξ for which $c'(\xi)$ is strictly negative. We have

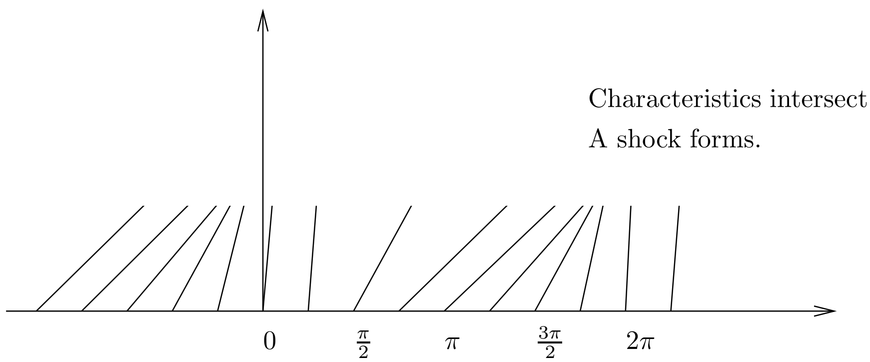
$$c'(\xi) = 2\varepsilon \sin(\xi)$$

Hence, a shock will occur ($c'(\xi) < 0,\ \forall \xi \in ((2k-1)\pi, 2k\pi)$). The first time it occurs is when (6) vanishes for the first time i.e.

$$t = \frac{-1}{\min c'(\xi)}$$

This is when $\xi = \frac{3\pi}{2}$, $t = \frac{1}{2\varepsilon}$ and $x = c(\frac{3\pi}{2})\frac{1}{2\varepsilon} + \frac{3\pi}{2}$. (We only really consider one period but of course the results can be extended by periodicity)

c increases in $(0, \pi)$ and decreases in $(\pi, 2\pi)$. We get roughly the following plot.



d)

Following the given hint, we look at the two characteristics starting at $\xi = \frac{3\pi}{2} \pm \theta$. They intersect at

$$x = c\big(\frac{3\pi}{2} + \theta\big)t + \frac{3\pi}{2} + \theta$$
$$x = c\big(\frac{3\pi}{2} - \theta\big)t + \frac{3\pi}{2} - \theta$$

Substracting the two equations, we get

$$2\varepsilon\big(\cos\big(\frac{3\pi}{2} + \theta\big)t - \cos\big(\frac{3\pi}{2} - \theta\big)t\big) = 2\theta$$

or

$$2\varepsilon \sin(\theta)t = \theta \tag{7}$$

For any $t \geq 0$, (7) has a unique positive solution θ (and $-\theta$ is the unique negative solution). Reciprocaly, if $\theta \neq k\pi$, (7) has a solution t, $t > 0$.

In addition, at $t = \frac{\theta}{2\varepsilon \sin(\theta)}$ and for any $\theta \neq k\pi$, the two characteristics intersect the same line

$$x = (1 - 2\rho_0)t + \frac{3\pi}{2} \tag{8}$$

We are now going to prove that the line defined by (8) is the curve where the shocks occur (the problem is 2π periodic and all the translations of this line are also shock lines).

We have

$$\begin{aligned}
\rho\Big(\frac{3\pi}{2} - \theta\Big) &= \rho_0 + \varepsilon \cos(\frac{3\pi}{2} - \theta) = \rho_0 - \varepsilon \sin(\theta) \\
\rho\Big(\frac{3\pi}{2} + \theta\Big) &= \rho_0 + \varepsilon \cos(\frac{3\pi}{2} + \theta) = \rho_0 + \varepsilon \sin(\theta)
\end{aligned}$$

hence

$$\rho\Big(\frac{3\pi}{2} - \theta\Big) + \rho\Big(\frac{3\pi}{2} + \theta\Big) = 2\rho_0 \tag{9}$$

The Rankine-Hugoniot condition is

$$\frac{ds}{dt} = \frac{(1 - \rho_l)\rho_l - (1 - \rho_r)\rho_r}{\rho_l - \rho_r} \tag{10}$$

where $x = s(t)$ is the equation of the shock curve and ρ_l (respectively ρ_r) denotes the value of ρ on the left (right) of the shock.

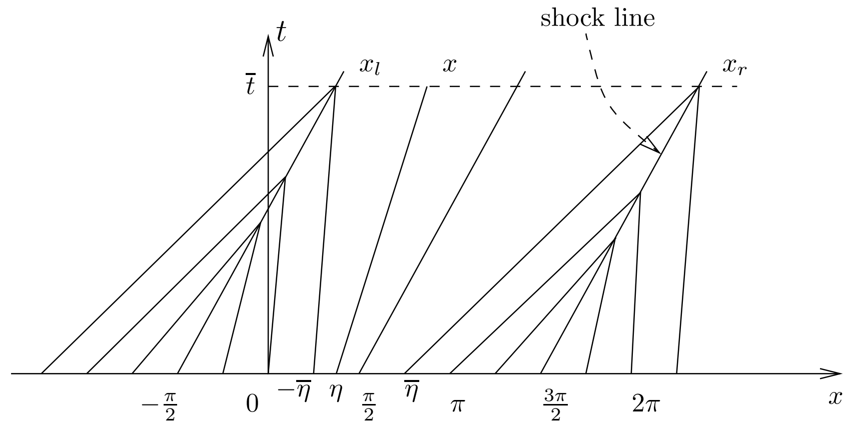
(10) yields

$$\frac{ds}{dt} = 1 - (\rho_l + \rho_r) \tag{11}$$

If we look at (8),

$$\begin{aligned}
\frac{dx}{dt} &= 1 - 2\rho_0 \\
&= 1 - \rho\Big(\frac{3\pi}{2} - \theta\Big) + \rho\Big(\frac{3\pi}{2} + \theta\Big) \quad \text{by (9)}
\end{aligned}$$

$x = (1 - 2\rho_0)t + \frac{3\pi}{2}$ therefore satisfies the Rankine-Hugoniot condition (10). It is a shock line. We have the following picture.

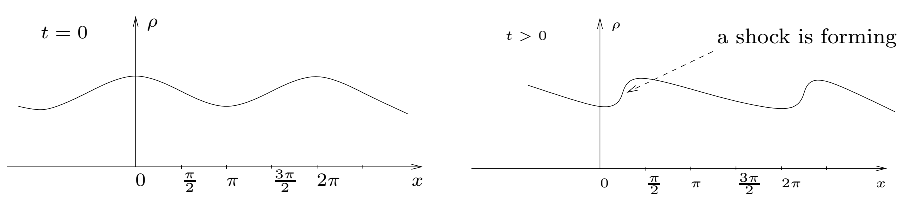


The height of the shock is

$$\rho_l - \rho_r = -2\varepsilon \sin(\theta)$$

when t tends to ∞, θ tends to 0 (see (7)). Hence

$$\lim_{t \to \infty} (\rho_l - \rho_r) = 0$$



We now investigate the behaviour of the solution as $t \to \infty$. We take a small strictly positive number $\overline{\eta}$ and look at the characteristics starting at $\frac{\pi}{2} \pm \overline{\eta}$ (see picture above). In (7), we take $\theta = \pi - \eta$ and get that the characteristics starting at $\frac{\pi}{2} + \eta$ intersect the characteristic starting at $\frac{3\pi}{2}$ at a time $\overline{t}$ given by

$$\overline{t} = \frac{\pi - \eta}{2\varepsilon \sin(\eta)} \tag{12}$$

We denote x_r the point where they intersect. One can check easily that the characteristic starting at $\frac{\pi}{2} - \overline{\eta}$ intersect the characteristic starting at $-\frac{\pi}{2}$ at the same time $\overline{t}$ as defined in (12) and at a position we denote x_l.

At $t = \overline{t}$ fixed, for any point belonging to $[x_l, x_r]$, there exists a unique $\eta \in [-\overline{\eta}, \overline{\eta}]$ such that the characteristic starting at $\frac{\pi}{2} + \eta$ contains $(x, \overline{t})$. We can prove that by showing that, when $\overline{\eta}$ is small,

$$x = c\left(\frac{\pi}{2} + \eta\right)\overline{t} + \frac{\pi}{2} + \eta$$

is a strictly decreasing function ($\frac{\partial x}{\partial \eta} < 0$) which maps $[-\overline{\eta}, \overline{\eta}]$ into $[x_l, x_r]$.

For $(x,t) \in [x_l, x_r] \times \{\overline{t}\}$ the solution is then given by

$$\rho = \rho_0 + \varepsilon \cos\left(\frac{\pi}{2} + \eta\right)$$
$$\text{and} \quad x = c\left(\frac{\pi}{2} + \eta\right)\overline{t} + \frac{\pi}{2} + \eta$$

(There is no shock in this region. The solution is given by using characteristics)

Since $|\eta| < \overline{\eta} \leq 1$, we expand these expressions and get

$$\rho = \rho_0 - \varepsilon\eta + O(\eta^3)$$
$$x = (1 - 2\rho_0 + 2\varepsilon\eta)\overline{t} + \frac{\pi}{2} + \eta + O(\eta^3)\overline{t}$$

Equation (12) can also be expanded and, after some calclution, gives an asymptotic relation between $\overline{t}$ and $\overline{\eta}$:

$$\overline{\eta} = \frac{\pi}{2\varepsilon\overline{t}} + O\left(\frac{1}{\overline{t}^2}\right)$$

Hence,

$$x = (1 - 2\rho_0 + 2\varepsilon\eta)\overline{t} + \frac{\pi}{2} + \eta + O\left(\frac{1}{\overline{t}^2}\right)$$

We express η in function of x

$$\eta = \frac{x - (1 - 2\rho_0)\overline{t} - \frac{\pi}{2}}{2\varepsilon\overline{t} + 1} + O\left(\frac{1}{\overline{t}^3}\right)$$
$$= \frac{x - (1 - 2\rho_0)\overline{t} - \frac{\pi}{2}}{2\varepsilon\overline{t}} + O\left(\frac{1}{\overline{t}^2}\right)$$

Finally,

$$\rho = \rho_0 - \frac{x - (1 - 2\rho_0)\overline{t} - \frac{\pi}{2}}{2\overline{t}} + O\left(\frac{1}{\overline{t}^2}\right)$$

When $\overline{t}$ tends to ∞, ρ looks like a straight line between two discontinuities, whose slope $(-\frac{1}{2\overline{t}})$ tends towards 0. We can compute ρ_l and ρ_r, the values of ρ on the left and right of the shock.

$$\rho_l = \rho_0 - \frac{\pi}{2\overline{t}} + O\left(\frac{1}{\overline{t}^2}\right)$$
$$\rho_r = \rho_0 + \frac{\pi}{2\overline{t}} + O\left(\frac{1}{\overline{t}^2}\right)$$

The density tends to ρ_0 and therefore when $\overline{t}$ tends to ∞, all the runners (that survive) go at the same speed $1 - \rho_0$.