Exercise Set 6

Problem 1

(a)

The scales can be justified as follows:

- $x^* = Lx$ — This gives $0 \le x \le 1$ in the problem, and there is no particular reason indicating that u^*, v^* and p^* undergo significant changes over smaller distances in the x^*-direction.
- $y^* = \varepsilon Ly$ — This gives $0 \le y \le 1$, and u^* changes from U to 0 over the interval $[0, \varepsilon L]$ in the y^*-direction.
- $u^* = Uu$ — Gives roughly $0 \le u \le 1$ in the problem.
- $v^* = \varepsilon Uv$ — Substituting $|\partial_{x^*} u^*| \sim U/L$ and $|\partial_{y^*} v^*| \sim V/(\varepsilon L)$, where V is the scale for v^* into the equation for conservation of mass one obtains $U/L \sim V/(\varepsilon L)$ and $V = \varepsilon U$ as the scaling factor for v^*.

(b)

With the additional scaling $p^* = Pp$ the system of equations becomes

$$\begin{aligned}
\frac{U^2}{L} u\partial_x u + \frac{U^2}{L} v\partial_y u &= -\frac{P}{L\rho}\partial_x p + \nu\Big(\frac{U}{L^2}\partial_{xx}u + \frac{U}{\varepsilon^2 L^2}\partial_{yy}u\Big) \\
\frac{U^2}{L} u\partial_x v + \frac{U^2}{L} v\partial_y v &= -\frac{P}{\varepsilon^2 L\rho}\partial_y p + \nu\Big(\frac{U}{L^2}\partial_{xx}v + \frac{U}{\varepsilon^2 L^2}\partial_{yy}v\Big) \\
\partial_x u + \partial_y v &= 0.
\end{aligned}$$

The term containing $1/\varepsilon^2$ is clearly the dominating term in the ν–term in the first equation. The assumption that the pressure term containing $p^*_{x^*}$ balances the viscous term leads to $P = \frac{\nu U \rho}{\varepsilon^2 L}$. Substituting this into the equations above leads to

$$\varepsilon\,\mathrm{Re}(u\partial_x u + v\partial_y u) = -\partial_x p + \varepsilon^2\partial_{xx}u + \partial_{yy}u, \tag{1}$$

$$\varepsilon^3\mathrm{Re}(u\partial_x v + v\partial_y v) = -\partial_y p + \varepsilon^4\partial_{xx}v + \varepsilon^2\partial_{yy}v, \tag{2}$$

$$\partial_x u + \partial_y v = 0. \tag{3}$$

(c)

If we try the value $\nu \approx 1\,\text{centiStokes} = 10^{-2}\,\text{cm}^2/\text{s} = 10^{-6}\,\text{m}^2/\text{s}$ together with the other given values we find $\mathrm{Re} \approx 100$. Only if the viscosity is an order of magnitude smaller, do we approach the critical limit $\mathrm{Re} \approx 1000$ – so it appears safe to assume that turbulence will not arise.

(d)

By setting ε to 0 in (1)–(3) one obtains

$$\partial_x p = \partial_{yy}u, \quad \partial_y p = 0, \quad \text{and} \quad \partial_x u + \partial_y v = 0. \tag{4}$$

Since p does not depend on y, the first equation gives

$$u = \frac{1}{2}y^2\partial_x p + Ay + B.$$

The boundary conditions $u(x,0)=1$ and $u(x,h(x))=0$ give $B=1$ and $A=-\frac{1}{2}h(x)\partial_x p-1/h(x)$ and thus

$$u=-\frac{1}{2}y(h-y)\partial_x p+1-\frac{y}{h}. \tag{5}$$

Integrating the third equation in (4) with respect to y and using the boundary conditions $v(x,0)=v(x,h(x))=0$ gives

$$0=\int_0^{h(x)}(\partial_x u+\partial_y v)dy=\int_0^{h(x)}\partial_x u\,dy+v(x,h(x))-v(x,0)=\int_0^{h(x)}\partial_x u\,dy,$$

and substituting $\partial_x u$ from (5) gives

$$\begin{aligned}0&=\int_0^{h(x)}\partial_x\big[-\tfrac{1}{2}y(h-y)\partial_x p+1-\frac{y}{h}\big]\,dy\\&=\partial_x\Big(\int_0^{h(x)}\big[-\frac{1}{2}y(h-y)\partial_x p+1-\frac{y}{h}\big]\,dy\Big)\\&=\partial_x\Big(-\frac{1}{12}(h^3\partial_x p)+\frac{1}{2}h\Big).\end{aligned}$$

(When we moved the derivative outside the integral sign we should have subtracted a term due to the x dependency in the upper limit of the integral, but this term is zero because the integrand is zero at $y=h$.) Integration gives $\frac{1}{6}h^3\partial_x p=h-\overline{h}$, where $\overline{h}$ is a constant of integration. This can be written $\partial_x p=6(h-\overline{h})/h^3$, which is Reynold's equation.

Finally, by setting $h=1-\alpha x$ one obtains

$$\partial_x p=6\left(\frac{1}{(1-\alpha x)^2}-\frac{\overline{h}}{(1-\alpha x)^3}\right).$$

Another integration using $p(0)=0$ gives

$$\begin{aligned}p(x)&=6\left[\frac{1}{\alpha(1-\alpha x)}-\frac{\overline{h}}{2\alpha(1-\alpha x)^2}\right]_0^x\\&=\frac{6}{\alpha(1-\alpha x)^2}\Big[(1-\alpha x)-\frac{\overline{h}}{2}-(1-\alpha x)^2+\frac{\overline{h}}{2}(1-\alpha x)^2\Big]\\&=\frac{6}{\alpha(1-\alpha x)^2}\Big(-\alpha x+\alpha^2x^2-\frac{\overline{h}}{2}(\alpha^2x^2-2\alpha x)\Big).\end{aligned}$$

Using $p(1)=0$ gives $0=-\alpha+\alpha^2-\frac{\overline{h}}{2}(\alpha^2-2\alpha)$ and therefore $\overline{h}=2(\alpha-1)/(\alpha-2)$.

(Note that $\overline{h}=\frac{2}{1/(1-\alpha)+1/1}$. Hence $\overline{h}$ is the harmonic mean of 1 and $1-\alpha$ (the height at $x=0$ and $x=1$ respectively), and thus $1-\alpha<\overline{h}<1$.)

The maximum pressure is attained at $h=\overline{h}$ since $\frac{\partial p}{\partial x}(h=\overline{h})=0$. We find

$$p_{\max}=\frac{3}{2}$$

Hence,

$$p^*_{\max}=\frac{3U\nu\rho}{2L\varepsilon^2}=30\text{ hPa}$$