Exercise Set 4

Problem 1

We are looking for the zeros of

$$36x^3 + (162 + 4\varepsilon)x^2 - 24\varepsilon x - 9\varepsilon = 0$$

We try $x = x_0 + \varepsilon x_1 + o(\varepsilon)$. At the first order we get

$$36x_0^3 + 162x_0^2 = 0$$

hence

$$x_0 = -\frac{9}{2} \quad \text{or} \quad x_0 = 0 \quad \text{(double root)}$$

At the second order, we get:

$$108x_0^2 x_1 + 4x_0^2 + 324x_0 x_1 - 24x_0 - 9 = 0$$

For $x_0 = -\frac{9}{2}$, we get

$$3645x_1 = 36$$

hence

$$x_1 = \frac{4}{405}$$

For $x_0 = 0$, we get $-9 = 0$ which is impossible and x cannot be expanded as $x = x_0 + \varepsilon x_1$ near 0 ($x - x_0$ is not of order ε). Let's try an other power of ε: $x = \varepsilon^p x_1 + o(\varepsilon^p)$. We get

$$36\varepsilon^{3p} x_1^3 + 162\varepsilon^{2p} x_1^2 + 4\varepsilon^{2p+1} x_1^2 - 24\varepsilon^{p+1} x_1 - 9\varepsilon = 0$$

$p = 1/2$ is the smallest strictly positive value which gives rise to more than one leading order term. We take $p = 1/2$ and it follows that

$$162x_1^2 = 9$$

and

$$x_1 = \pm\sqrt{\frac{9}{162}}$$

Finally, first approximations of the roots are given by

$$x = -\frac{9}{2} + \frac{4}{405}\varepsilon$$

and

$$x = \pm\sqrt{\frac{9}{162}}\sqrt{\varepsilon}$$

Problem 2

We want to solve

$$\varepsilon y'' + (1+x^2)y' + y = 0 \tag{1}$$

with

$$y(0) = 0,\ y(1) = 1$$

The inner solution y_m is given by

$$(1+x^2)y_m' + y_m = 0$$

which can be integrated explicitly

$$y_m = Ce^{-\arctan x}$$

C is a constant that we choose so that the boundary condition on the right is satisfied. We get

$$y_m = e^{\frac{\pi}{4} - \arctan x}$$

We have to determine the scaling x_l for the outer solution on the left

$$x_l = \frac{x}{\varepsilon^\alpha}$$

We have

$$y_l(x_l) = y(x)$$

and

$$\begin{aligned} y'(x) &= y_l'(x_l)\frac{1}{\varepsilon^\alpha} \\ y''(x) &= y_l''(x_l)\frac{1}{\varepsilon^{2\alpha}} \end{aligned}$$

Plugging that into equation (1), we get

$$\varepsilon^{1-2\alpha} y_l'' + (1+\varepsilon^{2\alpha}x_l^2)\varepsilon^{-\alpha} y_l' + y_l = 0 \tag{2}$$

The smallest α strictly bigger than zero which gives rise to more than one leading term is $\alpha = 1$ and then equation (2) yields

$$y_l'' + y_l' = 0$$

The general solution of this equation is

$$y_l = Ae^{-x_l} + B$$

The boundary condition $y(0) = 0$ implies that $B = -A$ and

$$y_l = A(1 - e^{-x_l})$$

To determine A, we match the outer and the inner expansions.

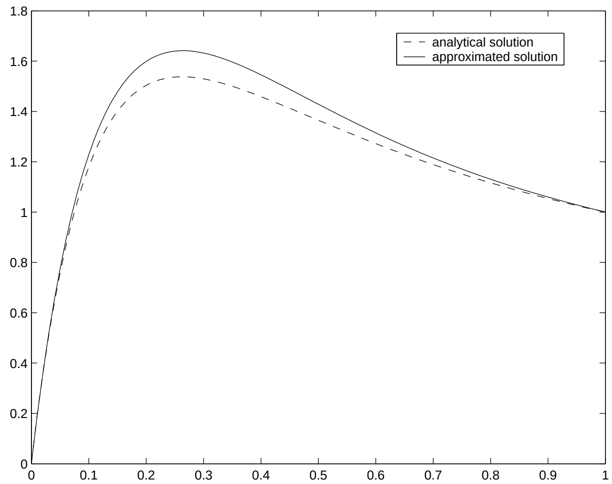
$$\lim_{x_l \to \infty} y_l(x_l) = \lim_{x \to 0} y_m(x)$$

hence

$$A = e^{\frac{\pi}{4}}$$

The total expansion is

$$\begin{aligned} y &= y_l(x_l) + y_m(x) - \lim_{x_l \to \infty} y_l(x_l) \\ &= e^{\frac{\pi}{4}} (e^{-\arctan x} - e^{-\frac{x}{\varepsilon}}) \end{aligned}$$



Approximated and analytical solution for $\varepsilon = 0.1$

Problem 3

If y satisfies the differential equation with the given boundary conditions, so does $y(1-x)$. The problem is symmetric with respect to $x = 1/2$ and we will focus our attention on the boundary layer on the left, around 0.

The inner expansion y_m of y satisfies

$$y_m'' + \lambda y_m = 0$$

which gives

$$y_m = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x \tag{3}$$

where A and B are constant.

Around 0, we rescale the problem with $x_l = \frac{x}{\varepsilon^\alpha}$ and $y_l(x_l) = y(x)$. We get

$$\varepsilon^{1-4\alpha} y_l'''' - \varepsilon^{-2\alpha} y_l'' = \lambda y_l$$

We take $\alpha = 1/2$ so that the fourth derivative is taken into account. At the lowest order, we have

$$y_l'''' - y_l'' = 0 \tag{4}$$

The general solution of this equation is

$$y_l = Ax_l + B + Ce^{x_l} + De^{-x_l}$$

The terms $x_l = x/\sqrt{\varepsilon}$ and $e^{x_l} = e^{\frac{x}{\sqrt{\varepsilon}}}$ are not possible, since we (implicitly) assumed while doing our expansion that $y_l = O(1)$. We are left with

$$y_l = B + De^{-x_l}$$

Taking into account the boundary conditions $y_l(0) = y_l'(0) = 0$, we get that $B = D = 0$ and

$$y_l = 0 \tag{5}$$

A similar calculation on the right, around $x = 1$, would give us

$$y_r = 0$$

we are now able to set the constants A and B in (3) by matching y_m with y_l and y_r. We have

$$\lim_{x \to 0} y_m(x) = \lim_{x_l \to \infty} y_l(x_l)$$

which implies, since $y_l = 0$,

$$A = 0$$

and

$$\lim_{x \to 1} y_m(x) = \lim_{x_r \to -\infty} y_r(x_r)$$

which implies

$$B \sin \sqrt{\lambda} = 0$$

We take $B \neq 0$ (we exclude the zero solution which is not a eigenfunction) and we arbitrarily set $B = 1$ since any multiple of a solution remains solution for the same eigenfrequency. We have:

$$\lambda = \pi^2 n^2, \quad n \in \mathbb{N} \backslash \{0\}$$

At the left-hand side, $y_l = 0$ does not give us a satisfactory picture of the solution. y_l has to grow at some point in order to match with $y_m = \sin \sqrt{\lambda} x$. The fact is that y_l is not only $O(1)$ but $O(\sqrt{\varepsilon})$ as we will now see. Let's introduce y_{l1} ($y_{l1} = O(1)$)defined as

$$y_{l1} = \sqrt{(\varepsilon)} y_{l1} + o(\sqrt{\varepsilon}) \tag{6}$$

y_{l1} satisfies (4) and we have, as earlier,

$$y_{l1} = Ax_l + B + Ce^{x_l} + De^{-x_l}$$

When ε tends to infinity, the term $x_l = x/\sqrt{\varepsilon}$ is compensated by the factor $\sqrt{\varepsilon}$ in front of y_{l1} in equation (6). Therefore A does not have to vanish. The term e^{x_l} however still goes to infinity and we must impose $C = 0$. The boundary conditions $y_{l1}(0) = y'_{l1}(0) = 0$ imply that $A = -B = D$ and y_{l1} takes the form

$$y_{l1} = A(x_l - 1 + e^{-x_l})$$

In order to match y_l with y_m, we introduce the intermediate scaling $\hat{x} = \varepsilon^{\beta} x_l = \varepsilon^{\beta-1/2}x$ where $\beta \in (0, \frac{1}{2})$. We rewrite y_l and y_m in terms of $\hat{x}$.

$$\begin{aligned} y_l &= A(\varepsilon^{1/2-\beta}\hat{x} - \varepsilon^{1/2} + \varepsilon^{1/2}e^{-\varepsilon^{1/2-\beta}\hat{x}}) \\ y_m &= \sin\left(\sqrt{\lambda}\varepsilon^{1/2-\beta}\hat{x}\right) = \sqrt{\lambda}\varepsilon^{1/2-\beta}\hat{x} + o(\varepsilon^{1/2-\beta}) \end{aligned}$$

We equal the lowest order terms and get

$$A = \sqrt{\lambda}$$

and the outer expansion on the left looks like

$$y_l = \sqrt{\lambda}(x - \sqrt{\varepsilon} + \sqrt{\varepsilon}e^{-\frac{x}{\sqrt{\varepsilon}}})$$

Problem 4

Let f denote

$$f(u) = u^2(u^2 - 1)$$

The fixed points of the systems are given by the roots of f.

$$u = \pm 1, 0$$

In general, for a given root u of f,

- if $f'(u) < 0$ then u is a stable point.
- if $f'(u) > 0$ then u is an unstable point.
- if $f'(u) = 0$ then we have to look at the second derivative. If $f''(u) \neq 0$, u is unstable otherwise we have to look at the next derivative and so forth.

These results are easily proved by looking at the taylor expansion of f around u. To illustrate this, let's look at $u = 0$ for our given function f. We have

$$\frac{du}{dt} = -u^2 + o(u^2)$$

and the system is unstable because if u is a little bit smaller than 0, the expansion above holds and du/dt is strictly negative, u decreases and therefore u goes further away from 0. In this case, we have

$$f'(0) = 0 \text{ and } f''(0) = -2$$

At $u = -1$,

$$f'(-1) = -2 < 0$$

and the system is stable (locally we have $d(u+1)/dt = -2(u+1) + o(u+1)$).

At $u = 1$,

$$f'(1) = 2 > 0$$

and the system is unstable.

Problem 5

This problem is revisited in the next exercise set (number 5) where the use of the theory presented in the course makes it much simpler.

The fixed point of the system are given by the zeros of f where

$$f(u) = u(9 - \mu u)(\mu + 2u - u^2)$$

case 1: $\mu > -1$

$\mu + 2u - u^2$ has two distinct zeros

$$u_1 = 1 - \sqrt{1+\mu} \text{ and } u_2 = 1 + \sqrt{1+\mu}$$

and the zeros (possibly multiple zeros) of f are

$$0, \ \frac{9}{\mu} \text{ (if } \mu \neq 0), \ u_1 \text{ and } u_2$$

At $u = 0$, $f'(0) = 9\mu$. If $\mu < 0$, 0 is stable. If $\mu > 0$, 0 is unstable. If $\mu = 0$, $f'(0) = 0$ but $f''(0) = 36 \neq 0$ and therefore 0 is unstable.

At $u = \frac{9}{\mu}$ ($\mu \neq 0$),

$$f'(\frac{9}{\mu}) = -\frac{9}{\mu^2}(\mu^3 + 18\mu - 81)$$

$\mu^3 + 18\mu - 81$ is strictly increasing from $-\infty$ to $+\infty$ and therefore has only one root that we denote μ_1. If $\mu < \mu_1$, $\frac{9}{\mu}$ is unstable while if $\mu > \mu_1$, $\frac{9}{\mu}$ is stable. If $\mu = \mu_1$, we have to look at $f''(\mu_1)$. After some calculation, we get that

$$f''(\frac{9}{\mu}) = -\frac{2}{\mu}(\mu^3 + 36\mu - 243)$$

for all $\mu \neq 0$. If we take $\mu = \mu_1$, since μ_1 satisfies

$$\mu_1^3 + 18\mu_1 - 81 = 0 \tag{7}$$

we have

$$\begin{aligned} \mu_1^3 + 36\mu_1 - 243 &= 81 - 18\mu_1 + 36\mu_1 - 243 \\ &= 18\mu_1 - 162 \\ &\neq 0 \end{aligned}$$

because $\mu_1 \neq 162/18$ (just check in (7)). Hence $f''(\frac{9}{\mu_1}) \neq 0$ and $9/\mu_1$ is unstable.

At $u = u_1$,

$$f'(u_1) = u_1(\mu u_1 - 9)(u_1 - u_2)$$

we have:

$$u_1 = 1 - \sqrt{1+\mu}$$

hence, if $\mu < 0$, $u_1 \in (0,1)$ and $f'(u_1) > 0$ and u_1 is unstable. If $\mu > 0$ then $f'(u_1) < 0$ (one has to check that $1-\sqrt{1+\mu} < \frac{9}{\mu}$ for all $\mu > 0$) and u_1 is stable. The case $\mu = 0$ gives $u_1 = 0$ and has already been investigated.

At $u = u_2$,

$$f'(u_2) = u_2(\mu u_2 - 9)(u_2 - u_1)$$

u_2 is always strictly positive as well as $u_2 - u_1$. $f'(u_2)$ vanishes when

$$u_2 = \frac{9}{\mu} \tag{8}$$

Equation (8) implies

$$1 + \mu = (\frac{9}{\mu} - 1)^2 \tag{9}$$

or

$$\mu^3 + 18\mu - 81 = 0$$

Therefore the only possible solution of (8) is μ_1. Conversly, μ_1 is indeed a solution of (8) because $9 > \mu_1$ ($\mu^3 + 18\mu - 81$ is stricly increasing and $9^3 + 18.9 - 81 > 0$) and therefore (9) implies (8). Thus, when $\mu = \mu_1$, $u_2 = 9/\mu_1$ and we have already investigated this case.

case 2: $\mu = -1$

We have a double root $u = 1$. Since $f''(1) = -20 \neq 0$, it corresponds to an unstable point. For $u = 0$ and $u = 9/\mu = -9$ the conlusions of the previous case remain unchanged.

case 3: $\mu < -1$

$\mu + 2u - u^2$ has no root. The only roots of f are $u = 0$ and $u = 9/\mu$. The stabilities of these points have already been given.

We sum up our results in the following table

μ		-1		0		μ_1	
0	stable	S	stable	U	unstable	U	unstable
$9/\mu$	unstable	U	unstable	$\|$	unstable	U	stable
u_1	undefined	U	unstable	U	stable	S	stable
u_2	undefined	U	stable	S	stable	U	unstable

where S and U stand respectively for stable and unstable.