# 75048 Mathematical Modeling 1998

Exercise 3 – Solution

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**Problem 1:**

**a)**

The scales can be justified as follows:

$x^* = Lx$ This gives $0 \le x \le 1$ in the problem, and there is no particular reason indicating that $u^*$, $v^*$ and $p^*$ undergo significant changes over smaller distances in the $x^*$-direction.

$y^* = \varepsilon L y$ This gives $0 \le y \le 1$, and $u^*$ changes from $U$ to 0 over the interval $[0, \varepsilon L]$ in the $y^*$-direction.

$u^* = Uu$ Gives roughly $0 \le u \le 1$ in the problem.

$v^* = \varepsilon U v$ Substituting $|\partial_{x^*} u^*| \sim U/L$ and $|\partial_{y^*} v^*| \sim V/(\varepsilon L)$, where $V$ is the scale for $v^*$ into the equation for conservation of mass one obtains $U/L \sim V/(\varepsilon L)$ and $V = \varepsilon U$ as the scaling factor for $v^*$.

**b)**

With the additional scaling $p^* = Pp$ the system of equations becomes

$$\begin{aligned}
\frac{U^2}{L} u \partial_x u + \frac{U^2}{L} v \partial_y u &= -\frac{P}{L\rho}\partial_x p + \nu\Big(\frac{U}{L^2}\partial_{xx} u + \frac{U}{\varepsilon^2 L^2}\partial_{yy} u\Big) \\
\frac{U^2}{L} u \partial_x v + \frac{U^2}{L} v \partial_y v &= -\frac{P}{\varepsilon^2 L\rho}\partial_y p + \nu\Big(\frac{U}{L^2}\partial_{xx} v + \frac{U}{\varepsilon^2 L^2}\partial_{yy} v\Big) \\
\partial_x u + \partial_y v &= 0.
\end{aligned}$$

The term containing $1/\varepsilon^2$ is clearly the dominating term in the $\nu$–term in the first equation. The assumption that the pressure term containing $p^*_{x^*}$ balances the viscous term leads to $P = \frac{\nu U \rho}{\varepsilon^2 L}$. Substituting this into the equations above leads to

$$\varepsilon \,\mathrm{Re}(u\partial_x u + v\partial_y u) = -\partial_x p + \varepsilon^2 \partial_{xx} u + \partial_{yy} u, \tag{1}$$

$$\varepsilon^3 \mathrm{Re}(u\partial_x v + v\partial_y v) = -\partial_y p + \varepsilon^4 \partial_{xx} v + \varepsilon^2 \partial_{yy} v, \tag{2}$$

$$\partial_x u + \partial_y v = 0. \tag{3}$$

**c)**

If we try the value $\nu \approx 1\,\text{centiStokes} = 10^{-2}\,\text{cm}^2/\text{s} = 10^{-6}\,\text{m}^2/\text{s}$ together with the other given values we find $\text{Re} \approx 100$. Only if the viscosity is an order of magnitude smaller, do we approach the critical limit $\text{Re} \approx 1000$ – so it appears safe to assume that turbulence will not arise.

**d)**

By setting $\varepsilon$ to 0 in (1)–(3) one obtains

$$\partial_x p = \partial_{yy} u, \quad \partial_y p = 0, \quad \text{and} \quad \partial_x u + \partial_y v = 0. \tag{4}$$

Since $p$ does not depend on $y$, the first equation gives

$$u = \frac{1}{2} y^2 \partial_x p + Ay + B.$$

The boundary conditions $u(x,0) = 1$ and $u(x,h(x)) = 0$ give $B = 1$ and $A = -\frac{1}{2}h(x)\partial_x p - 1/h(x)$ and thus

$$u = -\frac{1}{2} y(h-y)\partial_x p + 1 - \frac{y}{h}. \tag{5}$$

Integrating the third equation in (4) with respect to $y$ and using the boundary conditions $v(x,0) = v(x,h(x)) = 0$ gives

$$0 = \int_0^{h(x)} (\partial_x u + \partial_y v)dy = \int_0^{h(x)} \partial_x u\, dy + v(x,h(x)) - v(x,0) = \int_0^{h(x)} \partial_x u\, dy,$$

and substituting $\partial_x u$ from (5) gives

$$\begin{aligned} 0 &= \int_0^{h(x)} \partial_x \big[ -\tfrac{1}{2} y(h-y)\partial_x p + 1 - \tfrac{y}{h} \big]\, dy \\ &= \partial_x \Big( \int_0^{h(x)} \big[ -\frac{1}{2} y(h-y)\partial_x p + 1 - \frac{y}{h} \big]\, dy \Big) \\ &= \partial_x \Big( -\frac{1}{12}(h^3 \partial_x p) + \frac{1}{2} h \Big). \end{aligned}$$

(When we moved the derivative outside the integral sign we should have subtracted a term due to the $x$ dependency in the upper limit of the integral, but this term is zero because the integrand is zero at $y = h$.) Integration gives $\frac{1}{6} h^3 \partial_x p = h - \overline{h}$, where $\overline{h}$ is a constant of integration. This can be written $\partial_x p = 6(h - \overline{h})/h^3$, which is Reynold's equation.
Finally, by setting $h = 1 - \alpha x$ one obtains

$$\partial_x p = 6 \left( \frac{1}{(1-\alpha x)^2} - \frac{\overline{h}}{(1-\alpha x)^3} \right).$$

Another integration using $p(0) = 0$ gives

$$\begin{aligned}
p(x) &= 6\left[\frac{1}{\alpha(1-\alpha x)} - \frac{\overline{h}}{2\alpha(1-\alpha x)^2}\right]_0^x \\
&= \frac{6}{\alpha(1-\alpha x)^2}\Big[(1-\alpha x) - \frac{\overline{h}}{2} - (1-\alpha x)^2 + \frac{\overline{h}}{2}(1-\alpha x)^2\Big] \\
&= \frac{6}{\alpha(1-\alpha x)^2}\Big( -\alpha x + \alpha^2 x^2 - \frac{\overline{h}}{2}(\alpha^2 x^2 - 2\alpha x)\Big).
\end{aligned}$$

Using $p(1) = 0$ gives $0 = -\alpha + \alpha^2 - \frac{\overline{h}}{2}(\alpha^2 - 2\alpha)$ and therefore $\overline{h} = 2(\alpha-1)/(\alpha-2)$. (Note that $\overline{h} = \frac{2}{1/(1-\alpha)+1/1}$. Hence $\overline{h}$ is the harmonic mean of 1 and $1-\alpha$ (the height at $x = 0$ and $x = 1$ respectively), and thus $1 - \alpha < \overline{h} < 1$.)

## Problem 2:

### a)

Suppose the bar has heat conductivity $\lambda$ and heat capacity $c$. The heat flux is given by $-\lambda\nabla T^*$, and from energy conservation in a test volume (or area) $\Omega$ we get

$$\frac{d}{dt^*}\int_\Omega cT^*\,dV = \int_{\partial\Omega}\lambda\nabla T^*\cdot\boldsymbol{n}\,dS, \quad \text{or} \quad \int_\Omega c\frac{\partial T^*}{\partial t^*}\,dV = \int_\Omega \nabla\cdot(\lambda\nabla T^*)\,dV.$$

This holds in any test volume $\Omega$, so we conclude $c\,T^*_{t^*} = \lambda\,\triangle T^*$ (where $c$ and $\lambda$ are assumed to be constants). Note that we are seeking stationary solutions (this had been left out in the problem text), and thus $\triangle T^* = 0$. The assumption that the heat flux along the sides $y = \pm h$ is proportional to $T^* - T_0^*$ means

$$\mp\lambda\frac{\partial T^*}{\partial y^*} = K(T^* - T_0^*), \quad \text{for} \quad y^* = \pm h.$$

Using the scales

$$T = \frac{T^* - T_0^*}{T_1^* - T_0^*}, \quad x^* = Lx \quad \text{and} \quad y^* = hy = \varepsilon L y,$$

leads to

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 T}{\partial y^2} = 0.$$

The boundary conditions becomes $T(0, y) = 0$ and $T(1, y) = 1$, and for $y = \pm 1$ we get

$$\mp\lambda\frac{T_1^* - T_0^*}{\varepsilon L}\partial_y T = K(T_1^* - T_0^*)T,$$

giving $\mp\partial_y T = (\alpha\varepsilon)^2 T$ if we choose $(\alpha\varepsilon)^2 = \varepsilon KL/\lambda$, i.e., $\alpha = \sqrt{KL/(\varepsilon\lambda)}$.

### b)

Substituting $T = T_0 + \varepsilon^2 T_1 + \mathcal{O}(\varepsilon^4)$ into the equation gives

$$\partial_{yy}T_0 + \varepsilon^2(\partial_{xx}T_0 + \partial_{yy}T_1) = \mathcal{O}(\varepsilon^4), \tag{6}$$

and boundary conditions

$$T_0(0,y) + \varepsilon^2 T_1(0,y) = \mathcal{O}(\varepsilon^4) \tag{7}$$
$$T_0(1,y) + \varepsilon^2 T_1(1,y) = 1 + \mathcal{O}(\varepsilon^4) \tag{8}$$
$$\partial_y T_0 + \varepsilon^2(\partial_y T_1 \pm \alpha^2 T_0) = \mathcal{O}(\varepsilon^4) \qquad \text{for } y = \pm 1 \tag{9}$$

From $\varepsilon^0$ in (6) we derive $\partial_{yy}T_0 = 0$ and $T_0 = A(x)y + B(x)$. But (9) shows that $\partial_y T_0 = 0$ in $Y = \pm 1$, so $A(x) = 0$, and $T_0$ is a function of $x$ alone.
From the $\mathcal{O}(\varepsilon^2)$ term we get $\partial_{yy}T_1 + \partial_{xx}T_0$, and thus $T_1 = -\frac{1}{2}\partial_{xx}T_0 y^2 + C(x)y + D(x)$. For $y = \pm 1$ we get

$$0 = \partial_y T_1 \pm \alpha^2 T_0 = -\partial_{xx}T_0 y + C(x) \pm \alpha^2 T_0 = \pm(-\partial_{xx}T_0 + \alpha^2 T_0) + C(x).$$

We see that $C(x) = 0$ and therefore $\partial_{xx}T_0 = \alpha^2 T_0$. The latter of these two equations gives $T_0 = K_1 e^{\alpha x} + K_2 e^{-\alpha x}$. From the boundary condition $T_0(0) = 0$ we get $K_1 = -K_2$, and from $T_0(1) = 1$ we get $K_1(e^\alpha - e^{-\alpha}) = 1$, and thus

$$T_0 = \frac{e^{\alpha x} - e^{-\alpha x}}{e^\alpha - e^{-\alpha}} = \frac{\sinh \alpha x}{\sinh \alpha},$$

and we are done.

## Problem 3:

### a)

By Newton's second law of motion

$$Mp^*(t^*) - M\frac{u^*}{\tau} - F_{\text{air}} = M\frac{du^*}{dt^*},$$

where $F_{\text{air}}$ denotes air resistance. To determine an expression for $F_{\text{air}}$ we use dimensional analysis. It is reasonable to expect that $F_{\text{air}}$ depends on the air density $\rho_{\text{air}}$ and viscosity $\nu_{\text{air}}$, as well as the runner's speed $u^*$ and cross sectional area $A$ in the direction of motion. Moreover, we need a typical length scale $L$, which should be something like $\sqrt{A}$. This gives the dimensionless groups $\pi_1 = F_{\text{air}}/\rho_{\text{air}}A(u^*)^2$ and $\pi_2 = Lu^*/\nu_{\text{air}}$, the latter being the Reynold's number. Buckingham's $\pi$–theorem then gives $F_{\text{air}} = \rho_{\text{air}}(u^*)^2 A\psi(\text{Re})$. In fluid mechanics it is customary to write this as $F_{\text{air}} = \frac{1}{2}\rho_{\text{air}}(u^*)^2 AC_D(\text{Re})$, where $C_D$ is the drag coefficient. Thus,

$$M\frac{du^*}{dt^*} + M\frac{u^*}{\tau} + \tfrac{1}{2}\rho_{\text{air}}C_D(u^*)^2 A = Mp^*(t^*). \tag{10}$$

### b)

We are given that $\tau$ is a typical time constant, and it is reasonable to use this as time scale. The maximal acceleration $P$ provides a natural scale for $p^*$, leading to $P\tau$ as a scale for the velocity. Upon substitution in (10) we get

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \quad \varepsilon = \frac{1}{2}\rho_{\text{air}}C_D\tau^2 P\tfrac{A}{M}. \tag{11}$$

We find that the ratio of air resistance and inner resistance is approximately $\varepsilon u$ (if $p \approx u$ after scaling). By substituting the given numbers we get $\varepsilon \approx 0.03$ for a runner with mass 70-80 kg, and since $|u| \leq 1$ it seems fair to assume that air resistance is of little importance.

## c)

Substituting $u(t) = u_0(t) + \varepsilon u_1(t) + \mathcal{O}(\varepsilon^2)$ in (11) gives

$$\dot{u}_0 + \varepsilon \dot{u}_1 + \mathcal{O}(\varepsilon^2) + u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) + \varepsilon\big(u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)\big)^2 = p.$$

Collecting terms of the same order and using $u(0) = 0$ as initial condition gives us

$$\begin{aligned} \dot{u}_0 + u_0 &= p, & u_0(0) &= 0,\\ \dot{u}_1 + u_1 &= -u_0^2, & u_1(0) &= 0,\\ \dot{u}_2 + u_2 &= -2u_0u_1, & u_2(0) &= 0,\\ &\vdots & &\vdots \end{aligned}$$

The assumption $p^*(t^*) \equiv P$, i.e., $p(t) \equiv 1$ used in the two first equations gives

$$\begin{aligned} \dot{u}_0 + u_0 &= 1, & u_0(0) &= 0,\\ \dot{u}_1 + u_1 &= -u_0^2, & u_1(0) &= 0, \end{aligned}$$

which has solution $u_0(t) = 1 - e^{-t}$ and $u_1(t) = -1 + 2te^{-t} + e^{-2t}$ and thus

$$u(t) = 1 - e^{-t} + \varepsilon\big[-1 + 2te^{-t} + e^{-2t}\big] + \mathcal{O}(\varepsilon^2).$$

The runner reaches top speed when $u_0(t) \approx 1$. This happens approximately after $t = 3$, i.e., after $3\tau = 3$ seconds.

## d)

The air resistance term now becomes

$$F_{\text{air}} = \tfrac{1}{2}\rho_{\text{air}} C_D A\big(u^* - W\big)^2.$$

The scale for $W$ is $\tau P$ and we set $\delta = W/(\tau P)$. The top speed is reached when the acceleration is zero and is given by

$$U + \varepsilon\big(U - \delta\big)^2 = 1.$$

Using $U = U_0 + \varepsilon U_1 + \mathcal{O}(\varepsilon^2)$ results in the equations

$$U_0 = 1, \qquad U_1 + \big(U_0 - 1\big)^2 = 0,$$

and thus $U = 1 - \varepsilon(1-\delta)^2 + \mathcal{O}(\varepsilon^2)$.

## e)

We assume that Florence Griffith-Joyner ran at peak performance in all competitions. Moreover, we make the slightly doubtful assumption that she kept maximum speed the whole way. The maximal speed without wind is $U_0 = 1 - \varepsilon$, and $U_4 = 1 - 0.36\varepsilon$ is the maximal speed when the wind is $\delta = 0.4$. The time used is $T = 100/U$, and the ratio of the two times is $T_0/T_4 = (1 - 0.36\varepsilon)/(1-\varepsilon)$. To get an inkling about what this means in time we have to calculate 'her' $\varepsilon$. We assume she had the same acceleration, cross sectional area and drag coefficient as given in the problem text. Her mass was probably a little less, say 60 kg. This gives $\varepsilon \approx 0.03\frac{80}{60} = 0.04$. So with a tail wind of 4m/s we get $10.7 \cdot (1-0.04)/(1-0.04*0.36) = 10.42$ seconds. Not bad at all! (This is *too* accurate for our simple model.)