75048 Mathematical Modeling 1998

Exercise 2 – Solution

November 16, 1998

Problem 1:

Recall Fourier's law of heat conduction: The heat flux due to a temperature gradient is given by $-\kappa\nabla^* T^*$ where κ is the heat conductivity of the material in question. This leads to the usual heat conduction equation $\rho C\partial_{t^*}T^* = \kappa\Delta^* T^*$ where ρ is density and C is a specific heat capacity.[1] We add the initial condition $T^*(x^*, 0) = T_0$, assuming a constant initial temperature in the roast. Finally, we need a boundary condition to close the system. One one hand, the heat flux across the surface of the roast will be $\kappa\nabla^* T^* \cdot n = \kappa\partial_{n^*}T^*$ where n is the outward directed unit vector (thanks to the minus sign in Fourier's law, this is the *inward* heat flux). This must equal the term $\beta \cdot (T_1 - T^*)$ given in the problem.

We can thus write a model as follows, where Ω^* is the subset of space occupied by the roast, $\lambda = \kappa/(\rho C)$ is the *diffusivity* of heat in the roast, and $\gamma = \beta/\kappa$:

$$\begin{aligned} \partial_{t^*}T^* &= \lambda\Delta^* T^* \quad \text{in } \Omega^*, \\ T^*(x^*, 0) &= T_0 \quad \text{for } x^* \in \Omega^*, \\ \partial_{n^*}T^* &= \gamma \cdot (T^* - T_1) \quad \text{on } \partial\Omega^*. \end{aligned}$$

The proper SI units of κ are $(\mathrm{W/m^2})/(\mathrm{K/m}) = \mathrm{kg\,m\,K^{-1}s^{-3}}$, while those of the product ρC are $(\mathrm{kg/m^3}) \cdot (\mathrm{J/(kg\,K)}) = \mathrm{kg\,m^{-1}s^{-2}K^{-1}}$ – and so the units of $\lambda = \kappa/(\rho C)$ are $\mathrm{m^2/s}$, as it should be for a diffusivity. It follows that the first of the above equations is dimensionally correct. So is the second one, of course. And finally, the units of β are $(\mathrm{W/m^2})/\mathrm{K} = \mathrm{kg\,s^{-3}K^{-1}}$, so those of $\gamma = \beta/\kappa$ are $\mathrm{m^{-1}}$, and thus the last equation above is dimensionally correct.

To summarize, here are the relevant parameters and their units:

Parameter:	$T_1 - T_0$	$T_\mathsf{f} - T_0$	L	t_f^*	λ	γ
Units:	K	K	m	s	$\mathrm{m^2s^{-1}}$	$\mathrm{m^{-1}}$

where L is a typical length of the roast, T_f is the desired final center temperature of the roast, and t_f^* is the time it takes for the temperature to reach that value. Note that I only included temperature differences rather than absolute temperatures, since adding a constant to all the temperatures does not change the equations.

With six parameters involving three fundamental units we should get three independent dimensionless combinations:

$$\Theta = \frac{T_1 - T_0}{T_\mathsf{f} - T_0}, \quad \alpha = L\gamma, \quad \frac{t_\mathsf{f}^*\lambda}{L^2}.$$

[1] The stars indicate physical quantities that will later be nondimensionalized, as usual. ∇^* and Δ^* refer to the gradient and Laplace operators with respect to x^*.

By Buckingham's pi-theorem we should have a relationship of the form

$$t_{\mathsf{f}}^* = \frac{L^2}{\lambda} f(\alpha, \Theta). \tag{1}$$

This is not quite enough information for practical cooking, however. We now proceed to scale the equations. One natural length scale is L, which is derived from the roast itself (for example, use the radius of the roast). A corresponding time scale, derived from the heat equation itself, would be L^2/λ. This can be considered a rough measure of the time it takes heat to diffuse a distance L. It seems natural to measure non-dimensional temperature on a scale where T_0 corresponds to 0, and the desired final temperature T_{f} to 1. Hence we end up with the scalings

$$x^* = Lx, \quad t^* = \frac{L^2}{\lambda} t, \quad T^* = T_0 + (T_{\mathsf{f}} - T_0)T$$

and the model

$$\partial_t T = \Delta T \quad \text{in } \Omega, \tag{2}$$
$$T(x, 0) = 0 \quad \text{for } x \in \Omega, \tag{3}$$
$$\alpha T - \partial_n T = \alpha \Theta \quad \text{on } \partial\Omega. \tag{4}$$

Here, $\Theta > 1$ is a dimensionless measure of the oven temperature. In a typical cooking application, with $T_0 \approx 10\,°\mathrm{C}$ and $T_{\mathsf{f}} \approx 75\,°\mathrm{C}$, Θ will be somewhere in the range 2–4.

Let us look at a couple of special cases.

First, if $\alpha \gg 1$ it seems reasonable to ignore the $\partial_n T$ term in (4), leaving the condition in the form $T = \Theta$. Then the cooking time does not depend on α, and equation (1) implies that the cooking time is proportional to L^2, or in other words to $M^{2/3}$ where M is the mass of the roast (provided all roasts under consideration have the same general shape, and we are not varying the oven temperature).

Second, if $\alpha \ll 1$ we might reasonably ignore the αT term on the left-hand side of (4). But then the solution will depend linearly on $\alpha\Theta$, so (1) takes the form $t_{\mathsf{f}}^* = g(\alpha\Theta)L^2/\lambda$. We can say more in this case, however. Write $E = \int_\Omega T\, dV$. This is a non-dimensional measure of the total energy in the roast. We differentiate to find the rate of change of E:

$$\dot{E} = \int_\Omega \partial_t T\, dV = \int_\Omega \Delta T\, dV = \int_{\partial\Omega} \partial_n T\, dS = \alpha\Theta S$$

where S is the total (dimensionless!) surface area of the roast. Since a small value of α corresponds to a slow heating rate, we may assume that the temperature is fairly uniformly distributed in the roast, so that in particular $E \approx VT_{\mathsf{c}}$, where T_{c} is the center temperature, and V is the total (dimensionless) volume of the roast. Thus we get (after integrating) $T_{\mathsf{c}} \approx \alpha\Theta t S/V$, and this reaches its target value $T_{\mathsf{c}} = 1$ (corresponding to $T_{\mathsf{c}}^* = T_{\mathsf{f}}$) at $t_{\mathsf{f}} \approx V/(S\alpha\Theta)$, which corresponds to

$$t_{\mathsf{f}}^* = \frac{L^2 V}{S\alpha\Theta\lambda} = \frac{LV}{S\gamma\Theta\lambda}$$

You should keep in mind, however, that while L here is a real length, S and V are dimensionless quantities, and as such do not change if the roast is replaced by a larger, similarly shaped one. Hence we have a cooking time proportional to L – or equivalently, to $M^{1/3}$ in this case.

Parameters. Some of the parameters of the problem are easy to estimate. As a very rough estimate, we might use the density and thermal properties of water:

$$\rho = 1000\,\mathrm{kg\,m^{-3}}, \quad C = 4190\,\mathrm{J\,kg^{-1}K^{-1}}, \quad \kappa = 0.591\,\mathrm{W\,m^{-1}K^{-1}}.$$

From these values we find

$$\lambda = 0.141 \cdot 10^{-6}\,\mathrm{m^2s^{-1}}.$$

To estimate β we will have to dig much deeper into physics. *Planck's radiation law* states that a cavity whose walls are at a temperature T contains radiation with an energy density equal to

$$E(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}.$$

An integration yields

$$E = \int_0^\infty E(\nu)\,d\nu = \frac{8\pi^5k^4T^4}{15c^3h^3}$$

for the total radiation density per volume. This must be multiplied by $c/2$ to get the radiation energy flux. We conclude that a black body at temperature T radiates at a rate

$$\frac{Ec}{2} = \frac{4\pi^5k^4T^4}{15c^2h^3}$$

per unit area. It should follow that heat enters the surface of our roast at a rate

$$\frac{4\pi^5k^4(T_1^4-T^4)}{15c^2h^3} \approx \frac{16\pi^5k^4T_1^3(T_1-T)}{15c^2h^3}$$

which corresponds to

$$\beta \approx \frac{16\pi^5k^4T_1^3}{15c^2h^3} \approx 44\,\mathrm{W\,m^{-2}K^{-1}}$$

using $T_1 = 450\,\mathrm{K}$, and standard values of h, k, c from any table of physical constants. This gives the value

$$\gamma = \frac{\beta}{\kappa} \approx 74\,\mathrm{m^{-1}}, \quad \gamma^{-1} \approx 1.34\,\mathrm{cm}$$

Alternative scaling. The above calculations lead me to think that perhaps γ^{-1} is a more appropriate length scale in the problem, since we must expect the temperature to vary considerably over this interval. So the alternative scaling

$$x^* = \gamma^{-1}, \quad t^* = \frac{t}{\gamma^2\lambda}, \quad T^* = T_0 + (T_\mathsf{f} - T_0)T$$

yield the model on the form

$$\begin{aligned} \partial_t T &= \Delta T \quad \text{in } \Omega, \\ T(x,0) &= 0 \quad \text{for } x \in \Omega, \\ T - \partial_n T &= \Theta \quad \text{on } \partial\Omega. \end{aligned}$$

However, in this model the set Ω will be of a nondimensional size roughly equal to $L\gamma$.
If this is considerably larger than 1, as is likely for a large roast, it may be best to think of the heat conduction problem as a basically one-dimensional one; thus, we consider the heat penetrating into the roast from one surface. This would correspond to a model of the form (where $x = 0$ at the surface of the roast, and the positive x axis extends into the roast):

$$\begin{aligned} \partial_t T = \partial_x^2 T, &\qquad x > 0, t > 0; \\ T(x,0) = 0, &\qquad x > 0; \\ T(0,t) - \partial_x T(0,x) = \Theta, &\qquad t > 0. \end{aligned}$$

This model is a bit too difficult to solve directly. However, for small t we may expect the surface temperature $T(0,t)$ to be small, so we can use the simplified boundary condition $-\partial_x T(0,x) = \Theta$. For this problem, a similarity solution exists:

$$T(x,t) = \Theta \cdot \Big(\sqrt{t}e^{-x^2/4t} - x \operatorname{erfc} \frac{x}{2\sqrt{t}}\Big).$$

Before dimensionless time 1 has passed, the surface temperature has risen to near the oven temperature Θ, so from then on it might be better to use the boundary condition $T(0,t) = \Theta$. If we had used this from the beginning, we would also have a similarity solution

$$T(x,t) = \Theta \operatorname{erfc} \frac{x}{2\sqrt{t}}$$

The truth should be a temperature somewhat lower than this. In any case, this shows that the heat extends a distance on the order of $\sqrt{t}$ into the roast, which leads to the rough estimate that it takes a time given by $\sqrt{t_{\mathsf{f}}} \approx L\gamma$ to reach the center of the roast; thus $t_{\mathsf{f}} \approx L^2\gamma^2$, which is consistent with our first estimate.