75048 Mathematical Modeling 1998

Exercise 2 – Solution

October 18, 1998

Problem 1:

(We will return to this problem later.)

Problem 2:

Clearly one has the relation $\rho_0 l_0 = \rho l$. (Note that for thin bars, wires etc., the density is typically defined as mass per unit length.) From this we deduce that ρ_0 can be eliminated from the list of variables. For transversal vibrations, we expect the stretching of the string to be approximately constant – it is the *curvature* of the string combined with the force F which brings the string back towards its equilibrium. Hence we may eliminate even ρ_0 from our list of variables.[1] This leaves

	ω	l	ρ	F
M	0	0	1	1
L	0	1	-1	1
T	-1	0	0	-2

Since the matrix has rank 3 there is $4-3=1$ dimensionless variable. Buckingham's $\pi-$theorem yields a constant $\pi_1 = F/\rho l^2\omega^2$, or alternatively,

$$\omega = \frac{1}{\pi_1}\sqrt{\frac{F}{\rho l^2}}.$$

Substituting $F \approx k(l-l_0)$ (from the graph) gives

$$\omega \approx \frac{1}{\pi_1}\sqrt{\frac{k(l-l_0)}{l^2\rho_0 l_0^2/l}} = \frac{1}{\pi_1}\sqrt{\frac{k}{\rho_0 l_0^2}}\sqrt{1-\frac{l_0}{l}} \approx \frac{1}{\pi_1}\sqrt{\frac{k}{\rho_0 l_0^2}}\Big(1-\frac{l_0}{2l}\Big),$$

by the binomial formula. When $l > 2l_0$ the last factor does not change much as l increases, and the frequency will be approximatively constant. Just before the band reaches its limit the factor k will increase resulting in a slight increase in frequency.

[1] For longitudinal vibrations this would be different.

Problem 3:

(Problem 10, page 242 in Lin & Segel.) Substituting $x = \hat{x}g/V^2$ and $t = \hat{t}g/V$ into the equation

$$m\frac{d^2\hat{x}}{d\hat{t}^2} + k\frac{d\hat{x}}{d\hat{t}} = -mg, \qquad \hat{x}(0) = 0, \qquad \frac{d\hat{x}}{d\hat{t}}(0) = V, \tag{1}$$

yields

$$\ddot{x} + \beta\dot{x} + 1 = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = 1, \tag{2}$$

where $\beta = kV/mg$.

The problem is to find where and when the projectile reaches its maximum height, that is, determine t_m such that $\dot{x}(t_m) = 0$, og $x_m = x(t_m)$. Set $x(t) = x_0(t) + x_1(t)\beta + x_2(t)\beta^2 + \mathcal{O}(\beta^3)$. The calculations are straightforward and described in Lin & Segel. It is therefore carried out using Maple. It should be easy to recognize what is going on even without having heard of Maple. (Seeing what is possible might serve as motivation for learing more about Maple, though.)

```
> xeq:= diff(x(t), t$2) + beta*diff(x(t),t) + 1;
```

$$xeq := \left(\frac{\partial^2}{\partial t^2}\mathrm{x}(t)\right) + \beta\left(\frac{\partial}{\partial t}\mathrm{x}(t)\right) + 1$$

```
> with (powseries):
> powcreate(xt(n)='x_.n(t)');
> r:=tpsform(xt,beta,3);
```

$$r := \mathrm{x\_0}(t) + \mathrm{x\_1}(t)\,\beta + \mathrm{x\_2}(t)\,\beta^2 + \mathrm{O}\left(\beta^3\right)$$

```
> xeq2:=subs(x(t)=r,xeq):
> xeq2t:=taylor( xeq2, beta, 3);
```

$$xeq2t := \left(\left(\frac{\partial^2}{\partial t^2}\mathrm{x\_0}(t)\right) + 1\right) + \left(\left(\frac{\partial^2}{\partial t^2}\mathrm{x\_1}(t)\right) + \left(\frac{\partial}{\partial t}\mathrm{x\_0}(t)\right)\right)\beta + \left(\left(\frac{\partial^2}{\partial t^2}\mathrm{x\_2}(t)\right) + \left(\frac{\partial}{\partial t}\mathrm{x\_1}(t)\right)\right)\beta^2 + \mathrm{O}\left(\beta^3\right)$$

```
> deq1:=coeff(xeq2t,beta,0) = 0;
```

$$deq1 := \left(\frac{\partial^2}{\partial t^2}\mathrm{x\_0}(t)\right) + 1 = 0$$

```
> x_0s:=dsolve( {deq1, x_0(0)=0, D(x_0)(0)=1}, x_0(t));
```

$$x\_0s := \mathrm{x\_0}(t) = -\frac{1}{2}t^2 + t$$

```
> deq2:=subs(x_0s,coeff(xeq2t,beta,1))=0;
```

$$deq2 := \left(\frac{\partial^2}{\partial t^2}\mathrm{x\_1}(t)\right) + \left(\frac{\partial}{\partial t}\left(-\frac{1}{2}t^2 + t\right)\right) = 0$$

```
> x_1s:=dsolve({deq2, x_1(0)=0, D(x_1)(0)=0}, x_1(t));
```

$$x\_1s := \mathrm{x\_1}(t) = -\frac{1}{2}t^2 + \frac{1}{6}t^3$$

```
> deq3:=subs(x_0s,x_1s,coeff(xeq2t,beta,2))=0;
```

$$deq3 := \left(\frac{\partial^2}{\partial t^2}\mathrm{x\_2}(t)\right) + \left(\frac{\partial}{\partial t}\left(-\frac{1}{2}t^2 + \frac{1}{6}t^3\right)\right) = 0$$

```
> x_2s:=dsolve({deq3, x_2(0)=0, D(x_2)(0)=0}, x_2(t));
```

$$x\_2s := \mathrm{x\_2}(t) = -\frac{1}{24}t^4 + \frac{1}{6}t^3$$

```
> xm:=unapply( subs(x_0s,x_1s,x_2s,convert(r,polynom)), t);
```

$$xm := t \to -\frac{1}{2}t^2 + t + \left(-\frac{1}{2}t^2 + \frac{1}{6}t^3\right)\beta + \left(-\frac{1}{24}t^4 + \frac{1}{6}t^3\right)\beta^2$$

```
> powcreate(ts(n)='t_.n');
> rt:=tpsform(ts,beta,3);
```

$$rt := t\_0 + t\_1\,\beta + t\_2\,\beta^2 + \mathrm{O}\left(\beta^3\right)$$

```
> teq:=subs(t=rt,diff(xm(t),t)):
> teqs:=simplify(series( teq, beta, 3));
```

$$teqs := (-t\_0 + 1) + \left(-t\_1 - t\_0 + \frac{1}{2}t\_0^2\right)\beta + \left(-t\_2 - t\_1 + t\_0\,t\_1 - \frac{1}{6}t\_0^3 + \frac{1}{2}t\_0^2\right)\beta^2 + \mathrm{O}\left(\beta^3\right)$$

```
> t_0s:=t_0=solve(coeff(teqs,beta,0)=0, t_0);
```

$$t\_0s := t\_0 = 1$$

```
> t_1s:= t_1 = solve(subs(t_0s,coeff(teqs,beta,1)), t_1);
```

$$t\_1s := t\_1 = \frac{-1}{2}$$

```
> t_2s:= t_2 = solve(subs(t_0s, t_1s,coeff(teqs,beta,2)), t_2);
```

$$t\_2s := t\_2 = \frac{1}{3}$$

```
> tm := subs(t_0s,t_1s,t_2s,convert(rt,polynom));
```

$$tm := 1 - \frac{1}{2}\beta + \frac{1}{3}\beta^2$$

```
> series(xm(tm),beta,3);
```

$$\frac{1}{2} - \frac{1}{3}\beta + \frac{1}{4}\beta^2 + \mathrm{O}\left(\beta^3\right)$$

Problem 4:

a)

Here a provides a natural scale for the velocity. Assuming a is small compared to c gives eE_0/m_0 as scale for the acceleration, and also $a/(eE_0/m_0)$ as a scale for time. So,

$$\vec{v}^* = a\vec{v}, \qquad \text{og} \qquad t^* = \frac{am_0}{eE_0}t,$$

leading to

$$\frac{d}{dt}\frac{\vec{v}}{\sqrt{1-\epsilon|\vec{v}|^2}} = \vec{\imath}, \qquad \vec{v}(0) = \vec{\jmath}, \qquad \text{and} \quad \epsilon = \left(\frac{a}{c}\right)^2 \tag{3}$$

and we are done.

b)

The equation can easily be integrated once and we get

$$\frac{\vec{v}}{\sqrt{1-\epsilon|\vec{v}|^2}} = t\vec{\imath} + \frac{1}{\sqrt{1-\epsilon}}\vec{\jmath}.$$

Substitution of the given series yields

$$(\vec{v}_0 + \epsilon\vec{v}_1)\left(1 + \frac{\epsilon}{2}|\vec{v}_0|^2\right) = t\vec{\imath} + \left(1 + \frac{\epsilon}{2}\right)\vec{\jmath} + \mathcal{O}(\epsilon^2),$$

and by collecting terms we get

$$\begin{aligned}
\vec{v}_0 &= t\vec{\imath} + \vec{\jmath} \\
\vec{v}_1 &= -\frac{1}{2}(t + t^3)\vec{\imath} - \frac{1}{2}t^2\vec{\jmath}.
\end{aligned}$$