75048 Mathematical Modeling 1998

Exercise 1 – Solution

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Problem 1:

The quantities in this problem are diameter D, height h, sag δ, thickness d, density ρ, gravitational acceleration g and elasticity module E. Their dimensions are summarized below:

	D	h	d	δ	ρ	g	E
m	1	1	1	1	-3	1	-1
s	0	0	0	0	0	-2	-2
kg	0	0	0	0	1	0	1

The matrix clearly has rank 3 and consequently there are $7 - 3 = 4$ dimensionless variables. By inspection it readily follows that three of them may be chosen to be

$$\pi_1 = \frac{\delta}{D}, \quad \pi_2 = \frac{h}{D}, \quad \pi_3 = \frac{d}{D}.$$

The last dimensionless variable must contain the ratio of E by ρg. Since this ratio has dimension m it follows that one choice for π_4 is $\pi_4 = E/\rho g D$.

It now follows by Buckingham's pi-theorem (and the implicit function theorem) that

$$\frac{\delta}{D} = \Phi(\frac{h}{d}, \frac{d}{D}, \frac{E}{\rho g D}).$$

Problem 2:

The dimensions of time τ, pressure drop P, volume V, density ρ and viscosity μ are summarized below:

	P	τ	V	ρ	μ
m	-1	0	3	-3	-1
s	-2	1	0	0	-1
kg	1	0	0	1	1

Since the matrix has rank 3 there are $5 - 3 = 2$ dimensionless variables.

In order to get two dimensionless variables π_1, π_2 such that π_1 does not contain P and π_2 does not contain τ one has to find $\alpha_{1,2}$, $\beta_{1,2}$ and $\gamma_{1,2}$, such that

$$\pi_1 = \frac{\frac{1}{\tau}}{V^{\alpha_1}\rho^{\beta_1}\mu^{\gamma_1}}, \quad \text{and} \quad \pi_2 = \frac{P}{V^{\alpha_2}\rho^{\beta_2}\mu^{\gamma_2}}$$

are proper dimensionless variables. For π_1 this amounts to solving the set of equations

$$\begin{aligned} 3\alpha_1 - 3\beta_1 - \gamma_1 &= 0, \\ -\gamma_1 &= -1, \\ \beta_1 + \gamma_1 &= 0. \end{aligned}$$

This gives $\alpha_1 = -2/3$, $\beta_1 = -1$ and $\gamma_1 = 1$, and consequently, $\pi_1 = \rho V^{\frac{2}{3}}/\tau\mu$. In the same way one obtains $\pi_2 = P\rho V^{\frac{2}{3}}/\mu^2$.

Now, because we are given that there is a relation $f(P, \tau, V, \rho, \mu) = 0$ Buckingham's pi–theorem states that there is an equivalent relation $\Phi(\pi_1, \pi_2) = 0$, or (via the implicit function theorem)

$$\frac{P\rho V^{\frac{2}{3}}}{\mu^2} = \Psi\Big(\frac{\rho V^{\frac{2}{3}}}{\tau\mu}\Big).$$

Consequently, the variables on the $x-$ and $y-$axis are (most likely)

$$\log(\pi_1) = \log\left(\frac{\rho V^{\frac{2}{3}}}{\tau\mu}\right) \quad \text{and} \quad \log(\pi_2) = \log\left(\frac{P\rho V^{\frac{2}{3}}}{\mu^2}\right),$$

respectively.

Problem 3:

The quantities and dimensions in this problem are:

	ν	V	D	H	ρ	μ
m	0	1	1	1	-3	-1
s	-1	-1	0	0	0	-1
kg	0	0	0	0	1	1

Because the matrix has rank 3 there are $6 - 3 = 3$ dimensionless variables. By inspection $\pi_1 = D/H$ is clearly dimensionless. Likewise $\pi_2 = \nu D/V$ is dimensionless. The remaining dimensionless variable π_3 must contain the factor ρ/μ. Multiplying this by VD gives a fraction with dimension 1, thus $\pi_4 = \rho V D/\mu$. Since there is assumed to be a relation $g(\nu, V, D, H, \rho, \mu) = 0$, an application of Buckingham's pi-theorem gives an equivalent relation $h(\pi_1, \pi_2, \pi_3)$ which (via the implicit function theorem) can be written

$$\frac{\nu D}{V} = f\left(\frac{D}{H}, \frac{\rho V D}{\mu}\right).$$

In the 1:10 scale experiment (or in any experiment) the idea is that the dimensionless variables should be the same in the experiment as in the real application. Denoting model values with a superscript $^{\mathsf{M}}$ one has $\pi_i = \pi_i^{\mathsf{M}}$, $1 \le i \le 3$. Solving for V gives

$$V = \frac{\rho^{\mathsf{M}}}{\rho}\frac{D^{\mathsf{M}}}{D}\frac{\mu}{\mu^{\mathsf{M}}}V^{\mathsf{M}} = \frac{1000}{1}\cdot\frac{1}{10}\cdot\frac{2\cdot 10^{-5}}{10^{-3}}\cdot 7.5\ \mathrm{m\,s^{-1}} = 15\ \mathrm{m\,s^{-1}},$$

and similarly,

$$\nu = \frac{D^{\mathsf{M}}}{D}\cdot\frac{V}{V^{\mathsf{M}}}\cdot\nu^{\mathsf{M}} = \frac{1}{10}\cdot\frac{15}{7.5}\cdot 100\ \mathrm{s^{-1}} = 20\ \mathrm{s^{-1}}.$$