To måter, ett svar

$$
\begin{aligned}
&f(x+\Delta x, y+\Delta y) - f(x,y) \\
&\qquad = f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) + f(x+\Delta x, y) - f(x,y) \\
&\qquad = \int_0^{\Delta y} \frac{\partial f}{\partial y}(x+\Delta x, y+t)\,\mathrm{d}t + \int_0^{\Delta x} \frac{\partial f}{\partial x}(x+s, y)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&f(x+\Delta x, y+\Delta y) - f(x,y) \\
&\qquad = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y) \\
&\qquad = \int_0^{\Delta x} \frac{\partial f}{\partial x}(x+s, y+\Delta y)\,\mathrm{d}s + \int_0^{\Delta y} \frac{\partial f}{\partial y}(x, y+t)\,\mathrm{d}t
\end{aligned}
$$

Med andre ord:

$$\int_0^{\Delta y} \frac{\partial f}{\partial y}(x + \Delta x, y + t)\,\mathrm{d}t + \int_0^{\Delta x} \frac{\partial f}{\partial x}(x + s, y)\,\mathrm{d}s$$
$$= \int_0^{\Delta x} \frac{\partial f}{\partial x}(x + s, y + \Delta y)\,\mathrm{d}s + \int_0^{\Delta y} \frac{\partial f}{\partial y}(x, y + t)\,\mathrm{d}t$$

eller omskrevet:

$$\int_0^{\Delta x} \left[\frac{\partial f}{\partial x}(x + s, y + \Delta y) - \frac{\partial f}{\partial x}(x + s, y)\right] ds$$
$$= \int_0^{\Delta y} \left[\frac{\partial f}{\partial y}(x + \Delta x, y + t) - \frac{\partial f}{\partial y}(x, y + t)\right] dt$$

Og så integrerer vi igjen

$$\int_0^{\Delta x} \Big[\frac{\partial f}{\partial x}(x+s, y+\Delta y) - \frac{\partial f}{\partial x}(x+s, y)\Big]\, ds$$

$$= \int_0^{\Delta y} \Big[\frac{\partial f}{\partial y}(x+\Delta x, y+t) - \frac{\partial f}{\partial y}(x, y+t)\Big]\, dt$$

blir til

$$\int_0^{\Delta x} \Big[\int_0^{\Delta y} \frac{\partial^2 f}{\partial y \partial x}(x+s, y+t)\, dt\Big]\, ds$$

$$= \int_0^{\Delta y} \Big[\int_0^{\Delta x} \frac{\partial^2 f}{\partial x \partial y}(x+s, y+t)\, ds\Big]\, dt$$

Konklusjon

Vi har funnet

$$\int_0^{\Delta x}\Big[\int_0^{\Delta y}\frac{\partial^2 f}{\partial y\partial x}(x+s,y+t)\,dt\Big]\,ds = \int_0^{\Delta y}\Big[\int_0^{\Delta x}\frac{\partial^2 f}{\partial x\partial y}(x+s,y+t)\,ds\Big]\,dt$$

Men på grunn av (antatt) kontinuitet får vi

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\Delta x\,\Delta y \approx \frac{\partial^2 f}{\partial y\partial x}(x,y)\Delta x\,\Delta y$$

og hvis tilnærmingen er god nok (og det er den, i grensen når $\Delta x \to 0$ og $\Delta y \to 0$) så ender vi med

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y).$$