Exercise set 1

For TMA4230 Functional analysis

2006–01–23

Exercise 1.1. Show that l^1 is dense in c_0. Conclude from this that l^1 is not a closed subspace of l^∞. (Hint: If $x \in c_0$, consider the sequence $(x_1, x_2, \dots, x_n, 0, 0, \dots)$ and then let $n \to \infty$.)

Exercise 1.2. Recall that the *norm* of a linear map $T\colon X \to Y$, where X and Y are normed spaces, is defined as

$$\|T\| = \sup\{\|Tx\| \colon x \in X, \|x\| \le 1\}.$$

T is called *bounded* if $\|T\| < \infty$. It is called an *isomorphism* if it is bounded and has a bounded inverse. X and Y are called *isomorphic* if there exists an isomorphism between the two spaces.

Show that c and c_0 are isomorphic. (Hint: Consider the map $x \mapsto (x_\infty, x_1 - x_\infty, x_2 - x_\infty, \dots)$, where $x_\infty = \lim_{k\to\infty} x_x$.)

Exercise 1.3. Look at the definition of *extreme points* in the section on the Krein–Milman theorem (p. 59 in the present version of the notes).

The *closed unit ball* of a normed space X is the set of vectors $x \in X$ with $\|X\| \le 1$.

Show that if H is an inner product space then $x \in H$ is an extreme point of the unit ball of H if, and only if, $\|x\| = 1$. (Hint: Use the parallelogram law $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$.)

Show that $x \in c$ is an extreme point of the closed unit ball of c if, and only if, $|x_k| = 1$ for every k.

Show that the closed unit ball of c_0 has no extreme points.

Two normed spaces are called *isometric* if there exists an isometry between the two with norm 1; i.e., an isomorphism whose inverse also has norm 1. In other words, an isomorphism T with the property that $\|Tx\| = \|x\|$ for all x.

Show that an isometry maps extreme points of the unit ball of one space onto the extreme points of the unit ball of the other space, and conclude that c and c_0 are not isometric.