Suggested solutions for the midterm

given for TMA4230 Functional analysis

2006–03–17

Problem 1. *If X and Y are Banach spaces and $T\colon X \to Y$ is a bounded, surjective map then T is open.*

Problem 2. The canonical map $C\colon X \to X^{**}$ is given by $C(x) = \tilde{x}$, where $\tilde{x}(f) = f(x)$ for $f \in X^*$. X is called reflexive if C maps X *onto* X^{**}.

c_0 is not reflexive: Its second dual can be identified with ℓ^∞, and the canonical map is just the identity (or inclusion) map.

Problem 3. We find $\||u|^r\|_{p/r}^{p/r} = \int_\Omega |u|^{r(p/r)}\,d\mu = \|u\|_p^p$. and similarly $\||v|^r\|_{q/r}^{q/r} = \|v\|_q^q$. Also p/r and q/r are conjugate exponents $(1/(p/r) + 1/(q/r) = r/p + r/q = 1)$, so Hölder's inequality yields

$$\int_\Omega |u|^r |v|^r\,d\mu \le \||u|^r\|_{p/r}\,\||v|^r\|_{q/r} = \|u\|_p^r\,\|v\|_q^r.$$

Taking the $1/r$ power of this we get $\|uv\|_r \le \|u\|_p \|v\|_q$. Since this is finite, $uv \in L^r(\Omega)$. The inequality also shows that the mapping $u \mapsto uv$ is bounded with norm $\le \|v\|_q$. (This is actually an equality, but I didn't ask about that.)

Problem 4. The assumption $Ax \in \ell^\infty$ implies $|f_j(x)| \le \|Ax\|_\infty$ for each $x \in X$. Thus the family (f_j) of functionals is pointwise bounded, and hence uniformly bounded by the Banach–Steinhaus theorem (uniform boundedness principle). To use the theorem, we also need to know that each f_j is bounded, but this follows from $\sum_k |a_{jk}| < \infty$. In fact, $\|f_j\| = \sum_k |a_{jk}|$, by the standard duality: ℓ^1 is the dual of c_0. So our conclusion so far is

$$\sup_{j=1,2,\ldots} \|f_j\| = \sup_{j=1,2,\ldots} \sum_k |a_{jk}| < \infty$$

It follows that

$$\|Ax\|_\infty \le \sup_{j=1,2,\ldots} \|f_j\|\,\|x\|_\infty$$

for all $x \in c_0$, so that

$$\|A\| \le \sup_{j=1,2,\ldots} \|f_j\| = \sup_{j=1,2,\ldots} \sum_k |a_{jk}|$$

This inequality is actually an equality! This follows trivially from

$$\|A\| \ge \|f_j\| \qquad \text{for all } j,$$

which in its turn follows from

$$\|Ax\|_\infty \ge |(Ax)_j| = |f_j(x)|$$

and taking the supremum over all x with $\|x\|_\infty = 1$.

Alternatively, to prove that A is bounded you can use the closed graph theorem: It is not hard to show that the graph of A is closed, even without the assumption that it maps c_0 into ℓ^∞. For the graph of A is the intersection of all the sets $\{(x,y) \in c_0 \times \ell^\infty\colon f_j(x) = y_j\}$ for $j = 1,2,\ldots$, and they are all closed since f_j is bounded (and therefore continuous).