Suggested solutions for the midterm

given for TMA4230 Functional analysis

2005–03–17

Problem 1. *Any pointwise bounded set of bounded operators is uniformly bounded.* (See Kreyszig p. 249 for more details.) I accept an answer stating this for functionals rather than operators.

Problem 2. As the hint suggests, consider $f \in Y^*$ with $\|y\| \le 1$. Allow me to introduce the notation Y_1^* for this subset of Y. Then

$$|(T^* f)(x)| = |f(Tx)| \le \|Tx\| \qquad \text{for } x \in X,$$

so the set $\{T^* f \colon f \in Y_1^*\}$ is a pointwise bounded set of linear functionals. Since the members of this set are bounded by assumption, the Uniform boundedness theorem implies that the set is uniformly bounded. That is, there exists a constant M so that $\|T^* f\| \le M$ for all $f \in Y_1^*$.

In light of the above equality this means that $|f(Tx)| \le M\|x\|$ for all $x \in X$ and $f \in Y_1^*$. But from this $\|Tx\| = \sup_{f \in Y_1^*} |f(Tx)| \le M\|x\|$, so that indeed T is bounded with $\|T\| \le M$.

Problem 3. Hölder: $\displaystyle\int_\Omega |uv|\, d\mu \le \|u\|_p \|v\|_q$. In fact $\displaystyle\|u\|_p = \sup_{\|v\|_q = 1} \operatorname{Re} \int_\omega uv\, d\mu$.

Problem 4. As suggested in the hint, we start out with nonnegative f and u. Then $f * u$ is defined everywhere, though its value can be infinite at some points. It is not hard to see that when $u \ge 0$, the norm equality in problem 3 is still true where we take the supremum only over $v \ge 0$. So we consider any $v \in L^q$ with $v \ge 0$:

$$\int_{\mathbb{R}} (f*u)(x)v(x)\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)u(x-t)v(x)\,dt\,dx = \int_{\mathbb{R}} f(t) \underbrace{\int_{\mathbb{R}} u(x-t)v(x)\,dx}_{\le \|u\|_p\|v\|_q}\,dt \le \|f\|_1\|u\|_p\|v\|_q$$

where I have used the fact that the translated function $x \mapsto u(x-t)$ belongs to L^p with the same norm as u itself, and applied Hölder to the inner integral. Thus

$$\|f*u\|_p = \sup_{\|v_q = 1\|} \int_{\mathbb{R}} (f*u)(x)v(x)\,dx \le \|f\|_1\|u\|_p.$$

In general, when f and u are not necessarily nonnegative, note that the above gives us $\||f| * |u|\|_p \le \|f\|_1\|u\|_p$. In particular, $|f| * |u| < \infty$ almost everywhere. And at any such point, the integral defining $f * u$ converges, and $|f * u| \le |f| * |u|$. So $\|f * u\|_p \le \||f| * |u|\| \le \|f\|_1\|u\|_p$.

Problem 5. From the introductory comments to the problem we conclude that the almost periodic sequences form a closed subspace A of ℓ^∞.

Moreover $e_1 \notin A$. This requires proof: If x is periodic with period p, then in particular $x_{p+1} = x_1$. Since either $|x_1| \ge \frac{1}{2}$ or $|x_1 - 1| \ge \frac{1}{2}$, it follows from a look at the first and $p+1$st components that $\|x - e_1\| \ge \frac{1}{2}$. This inequality proves our claim.

The existence of a bounded linear functional f with $\|f\| = 1$, which vanishes on A and satisfies $f(e_1) = \frac{1}{2}$, is now a well known consequence of the Hahn–Banach theorem. (See Kreyszig Lemma 4.6-7, on p. 243.)