Solution set 5

to some problems given for TMA4230 Functional analysis

2004–03–09

Exercise A.1. I will not write up the full solutions to all of these simple exercises.

To prove $\bigcup\{C^{\mathsf{c}}\colon C \in \mathcal{C}\} = \left(\bigcap \mathcal{C}\right)^{\mathsf{c}}$, note that $x \in \bigcup\{C^{\mathsf{c}}\colon C \in \mathcal{C}\}$ means $x \in C^{\mathsf{c}}$ for every $C \in \mathcal{C}$, which means $x \notin C$ for every $C \in \mathcal{C}$, which means $x \in C$ for *no* $C \in \mathcal{C}$, which is the opposite of $x \in C$ for some $C \in \mathcal{C}$, which means $x \notin \bigcap \mathcal{C}$.

And to prove $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, note that $x \in f^{-1}(A \setminus B)$ means $f(x) \in A \setminus B$, which means $f(x) \in A$ but $f(x) \notin B$, which is the same as $x \in f^{-1}(A)$ but $x \notin f^{-1}(B)$, which is the same as $x \in f^{-1}(A) \setminus f^{-1}(B)$.

Exercise A.2. Remember that f is continuous if and only if $f^{-1}(V)$ is open for every open $V \subseteq Y$, and note that taking complements establishes a one-to-one correspondence between open sets and closed sets. So f is continuous if and only if $f^{-1}(V)^{\mathsf{c}}$ is closed for every open $V \subseteq Y$, which is equivalent to $f^{-1}(V^{\mathsf{c}})$ being closed for every open $V \subseteq Y$, which is equivalent to $f^{-1}(F)$ being closed for every closed $F \subseteq Y$ (where we replaced V by F^{c} in the last step).

Exercise A.3. First, assume that K is compact with the original definition. Let $\mathcal{V}$ be an open cover of K in the sense of the last paragraph of the exercise. Then let $\mathcal{U} = \{K \cap V\colon V \in \mathcal{V}\}$. Each member of U is a (relatively) open subset of K, and $\bigcup \mathcal{U} = K \cap \bigcup V = K$, since $\bigcup V \supseteq K$. Thus $\mathcal{U}$ is an open cover of K in the original definition. By compactness of K, it has a finite subcover $\{K \cap V_1, \ldots, K \cap V_n$ with $V_1, \ldots, V_n \in \mathcal{V}$. So $(K \cap V_1) \cup \ldots \cup (K \cap V_n) = K$, which implies $\{V_1 \cup \ldots \cup V_n \supset K$, and one implication is proved.

Conversely, if K satisfies the condition of the last paragraph of the exercise, let $\mathcal{U}$ be an open cover of K as originally defined. Let $\mathcal{V}$ consist of all open subsets $V \subseteq X$ so that $K \cap V \in \mathcal{U}$. Since every $U \in \mathcal{U}$ is open, it can be written $K \cap V$ for at least one open set $V \subseteq X$, and so $\mathcal{V}$ is an open cover (new definition) for K. Pick a finite subcover (new definition) $\{V_1, \ldots, V_n\}$. Then $\{K \cap V_1, \ldots, K \cap V_n\}$ is a finite subcover (original definition) of K.

Exercise A.4. Let K be compact, and let $L \subseteq K$ be closed. Any collection of closed subsets of L is also a collection of closed subsets of K, and if it has the finite intersection property, it has nonempty intersection because K is compact. This shows that L is compact.

Exercise A.5. Let X be Hausdorff and K a compact subset of X.

If $x \in \overline{K}$, there is a filter $\mathcal{F}$ in K converging to x. (In fact, one can take $\mathcal{F}$ to be generated by the sets $K \cap V$ for neighbourhoods V of x.) Since K is compact, there is a finer filter $\mathcal{G}$ which converges in K, to a point $y \in K$ This filter then converges to y in X as well, and it converges to x too, since it refines $\mathcal{F}$. But no filter can have two distinct limits on a Hausdorff space, so $x = y \in K$. Thus K is closed. (There are many different ways to organize this proof, not all of which use filters.)

Exercise A.6.

(a) First, if $x \neq y$ then the Hausdorff property means that x and y have disjoint neighbourhoods in the original topology. But then these are neighbourhoods in the stronger topology as well, and the Hausdorff property in the stronger topology follows.

Now let $F \subseteq X$ be closed in the stronger topology, but not in the original topology. If X is compact in the stronger topology, then so is F (exercise A.4). But then F is compact in the original topology as well, by the easy part of (b) below. This implies that F is closed in the original topology (exercise A.5), and this is a contradiction.

(b) First, consider an open (in the weaker topology) cover of X. Then this is an open (in the original topology) cover as well, and by compactness in the original topology, it has a finite subcover. So X is compact in the weaker topology.

Now let $F \subseteq X$ be closed in the original topology, but not in the weaker topology. Consider an open cover of F (in the weaker topology, and in the sense of exercise A.3). This is then also an open cover of F in the original topology, so it has a finite subcover. This shows that F is in fact compact in the weaker topology. If the weaker topology is Hausdorff, then F must be closed in the weaker topology (exercise A.5), and this is a contradiction.

Exercise A.7. By definition, the constructed collection $\mathcal{T}$ contains X as a member, and arbitrary unions of elements of $\mathcal{T}$ belong to $\mathcal{T}$. To show that $\mathcal{T}$ is a topology, it only remains to show that $U \cap V \in \mathcal{T}$ when $U, V \in \mathcal{T}$. So write $U = \bigcup \mathcal{U}$ where $\mathcal{U} \subseteq \mathcal{B}'$, and $V = \bigcup \mathcal{V}$ where $\mathcal{V} \subseteq \mathcal{B}'$. Then

$$U \cap V = \left(\bigcup \mathcal{U}\right) \cap \left(\bigcup \mathcal{V}\right) = \bigcup_{U \in \mathcal{U}} \left(U \cap \bigcup \mathcal{V}\right) = \bigcup_{U \in \mathcal{U}} \bigcup_{V \in \mathcal{V}} U \cap V \in \mathcal{T},$$

because each U and each V in the union belongs to $\mathcal{B}'$, and so $U \cap V \in \mathcal{B}'$ as well.

Because $\mathcal{T}$ is created from $\mathcal{B}$ by taking finite intersections and arbitrary unions, it is clear that any topology containing $\mathcal{B}$ must contain $\mathcal{T}$ as well. Thus $\mathcal{T}$ is the weakest topology containing $\mathcal{B}$, as stated.

My definitions of *base* for a topology was wrong. (Even the word was wrong: It should be *base*, not basis.) A base $\mathcal{B}$ for a topology $\mathcal{T}$ is a subset of $\mathcal{T}$ so that every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$. In the above situation, $\mathcal{B}'$ is a base for $\mathcal{T}$, but $\mathcal{B}$ need not be. This has no consequences for the next problem, for $\mathcal{B}'$ is countable if $\mathcal{B}$ is countable.

Exercise A.8. In a metric space X, any point x has the countable filterbase consisting of balls $B_{1/n}(x)$, for $n = 1, 2, \ldots$.

If $\mathcal{T}$ has a countable base $\mathcal{B}$ and $x \in X$, then $\{V \in \mathcal{B} \colon x \in V\}$ is a countable base for the neighbourhood filter at x, so X is first countable.

It is tempting to look for a metrizable space that is not second countable, since metric spaces are automatically first countable. In fact we may try using the discrete topology on some set X, which is first countable in the extreme ($\{\{x\}\}$ is a base for the neighbourhood filter at x). If X is uncountable, then surely the discrete topology on X cannot be second countable?

This seems to require a bit more set theory than I had realized when I posed this question, so I am afraid I have been a bit unfair.

Let us go a bit further and let $X = \mathbb{R}$. Remember, we are using the *discrete* topology on X – I am chosing the set $\mathbb{R}$ only because this set has lots of elements. A topology $\mathcal{T}$ with a countable basis $\mathcal{B}$ has a cardinality at most equal to that of $\mathbb{R}$, since it is formed by unions of members of the countable set $\mathcal{B}$, and the set of subsets of a countably infinite sets has cardinality equal to that of $\mathbb{R}$. But the discrete topology on X is the set of *all* subsets of X, and that has greater cardinality than X itself, and hence greater cardinality to $\mathcal{T}$. So the discrete topology on $\mathbb{R}$ is not second countable.

What is cardinality? Recall that any set X can be wellordered. With each such wellorder, X becomes order isomorphic to some ordinal number. The cardinality of X is the smalles ordinal number one can get in this way. A *cardinal number* is an ordinal number which is the cardinality of some set.

For example, every finite ordinal number is a cardinal number, and so is ω, the smalles infinite ordinal. ω is the cardinality of the natural numbers. When we think of ω as a cardinal number, we also write it as $\aleph_0$.[1] The immediate successors $\omega + 1$, $\omega + 2$, …, $\omega + \omega$ and so forth are not cardinal numbers, since they are all countable. The smallest cardinal number bigger than $\aleph_0$ is called $\aleph_1$, and so forth and so on.

Cantor's theorem states that the set $\mathcal{P}(X)$ of subsets of a given set X has strictly greater cardinality than X itself. In particular the cardinality of $\mathcal{P}(\mathbb{N})$ is greater than $\aleph_0$. What used to be a famous conjecture is that the cardinality of $\mathcal{P}(\mathbb{N})$, also written as $2^{\aleph_0}$, equals $\aleph_1$. This so-called *continuum hypothesis* has later been shown to be independent of the other axioms of set theory, as has the *generalized continuum hypothesis*, that $2^{\aleph_n} = \aleph_{n+1}$ for every ordinal n.

The proof of Cantor's theorem is by the famous *diagonal argument*: Assume that $f \colon X \twoheadrightarrow \mathcal{P}(X)$ maps X onto $\mathcal{P}(X)$. Form the set $D = \{x \in X : x \notin f(x)$. Now if $D = f(w)$ for some $w \in X$, then $w \in D \Leftrightarrow w \notin D$, a contradiction.

[1] $\aleph$ is the first letter of the Hebrew alphabet. A vowel, in a written language without vowels? I am not sure I understand this.