Solution set 4

to some problems given for TMA4230 Functional analysis

2004–02–24

Problem 4.12.2. There are many ways to construct an open mapping which does not map closed sets to closed sets. For eksample, consider the projection map onto the first coordinate in $\mathbb{R}^2$, in other words $f(x, y) = x$. It maps the closed set $\{(x, y)\colon xy \ge 1,\ x > 0\}$ to the non-closed set $x\colon x > 0$.

Problem 4.12.5. In a *very* brief notation, we write the definition of T as $T(x)_j = x_j/j$.[1] Clearly, the inverse map is given by $T^{-1}(y)_j = jy_j$. Both T and T^{-1} map sequences with only finitely many non-zero entries to sequences of the same kind. Also T is linear, since

$$T(x+y)_j = (x+y)_j/j = x_j/j + y_j/j = T(x)_j + T(y)_j, \quad T(\alpha x)_j = \alpha x_j/j = \alpha T(x)_j.$$

T is bounded, since $\|Tx\| = \sup_j |(Tx)_j| = \sup_j |x_j/j| \le \sup_j |x_j| = \|x\|$. But T^{-1} is unbounded, for if e_n is the sequence which is 1 in the nth position and 0 elsewhere, then $Te_n = ne_n$, so $\|Te_n\| = n\|e_n\|$.

This does not contradict the open mapping theorem because X is not complete.

Problem 4.12.6. If $\mathcal{R}(T)$ is closed, then it is complete, since Y is complete. By the open mapping theorem (with Y replaced by $\mathcal{R}(T)$), the inverse is bounded.

Conversely, if T^{-1} is bounded on $\mathcal{R}(T)$ then $\mathcal{R}(T)$ is complete, since it is isomorphic to the complete space X. *In detail*: Let (y_k) be a Cauchy sequence in $\mathcal{R}(T)$. Write $y_k = Tx_k$ with $x_k = T^{-1}y_k$. Since $\|x_j - x_k\| = \big\|T^{-1}(y_j - y_k)\big\| \le \|T^{-1}\|\|y_j - y_k\|$, the sequence (x_k) is also Cauchy, and hence convergent. But if $x_k \to z$ then $y_k = Tx_k \to Tz$ since T is bounded. This proves that $\mathcal{R}(T)$ is complete.

But a complete subspace of any metric space is closed, and this completes the proof. *In detail*: If a sequence in the subspace converges to a point in the full space, then that sequence is Cauchy, and hence convergent in the subspace. So the limit is in the subspace.

Problem 4.12.8. If there is no number b with $\|x\|_2 \le b\|x\|_1$ for all x, then there is no upper bound for the fractions $\|x\|_2/\|x\|_1$. So for each n we can find x_n with $\|x_n\|_2/\|x_n\|_1 > n$. This inequality is unaffected if we replace x_n by a scalar multiple of itself, so we may assume $\|x_n\|_2 = 1$. Then $\|x_n\|_1 < 1/n$, so $\|x\|_1 \to 0$, but $\|x_n\|_2 \not\to 0$. This contradicts the assumption.

Let $T\colon X_1 \to X_2$ be the identity map: $Tx = x$. (X_1 and X_2 are really the same space, but we use different norms.) What we proved above is that T is bounded. Since it is clearly invertible, the inverse is also bounded. This may not seem like a big surprise, since $T^{-1} = T$. But the role of the two norms are exchanged so we now have $\|x\|_1 \le \|T^{-1}\|\|x\|_2$. With $a = \|T^{-1}\|^{-1}$ we get $a\|x\|_1 \le \|x\|_2$.

To say $x_n \to z$ in X_1 is to say $\|x_n - z\|_1 \to 0$. But since $\|x_n - z\|_2 \le b\|x_n - z\|_1$, we get $x_n \to z$ in X_2. The converse implication, that convergence in X_2 implies convergence in X_1, is proved the same way.

Problem 4.13.7. If X and Y are Banach spaces and $T\colon X \to Y$ is bounded and bijective, then T has a closed graph because it is bounded. But there is a natural isomorphism $X \times Y \to Y \times X$ which maps (x, y) to (y, x). This isomorphism is isometric, and it maps the graph of T to the graph of T^{-1}. Thus T^{-1} has a closed graph. But then T^{-1} is bounded, by the closed graph theorem.

Problem 4.13.11. Let $T\colon X \to Y$ be closed and linear. (Note that T need not be bounded, as X and Y have not been assumed to be complete.) Now $\mathcal{N}(T) \times \{0\} = \mathcal{G}(T) \cap (X \times \{0\})$ is closed, since $\mathcal{G}(T)$ is closed by assumption and $X \times \{0\}$ is also closed in $X \times Y$. Thus $\mathcal{N}(T)$ is closed, being the inverse image of $\mathcal{N}(T) \times \{0\}$ under the mapping $X \to X \times Y$ given by $x \mapsto (x, 0)$.

Problem 4.13.12. This can be solved by considering a sequence in the graph of $T_1 + T_2$ explicitly, but here is (perhaps) a more elegant solution:

We note that an element in the graph of T_1+T_2 is written $(x, T_1x+T_2x) = (x, y+T_2x)$ where $(x, y) \in \mathcal{G}(T_1)$. We define a linear operator $S\colon X \times Y \to X \times Y$ by $S(x, y) = (x, y + T_2x)$. Then S is clearly bounded, and it has a bounded inverse $S^{-1}(x, y) = (x, y - T_2x)$. The isomorphism S maps the graph of T_1 to the graph of $T_1 + T_2$. Thus, if the former is closed then so is the latter.

Problem 4.13.13. Let $y_n \in \mathcal{R}(T)$ with $y_n \to w$. Then $y_n = Tx_n$ where $x_n = T^{-1}y_n$. Since T^{-1} is bounded, (x_n) is a Cauchy sequence. Since X is a Banach space, $x_n \to z$ for some $z \in X$. Then $(x_n, y_n) \to (z, w)$ in $X \times Y$. Since $(x_n, y_n) \in \mathcal{G}(T)$ and $\mathcal{G}(T)$ is closed, $(z, w) \in \mathcal{G}(T)$, so $w = Tz$. In particular, $w \in \mathcal{R}(T)$. We have proved that $\mathcal{R}(T)$ is closed.

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[1] I denote the jth component of x by x_j as usual.