Exercise set A

Some exercises for TMA4230 Functional analysis

2004–02–26

Exercise A.1. (This is just a warmup exercise, in case these results are not familiar. Skip it if you don't feel you need to do it.) Whenever we work within a fixed set X, it is convenient to write the complement of a subset $A \subseteq X$ as $A^{\mathsf{c}} = X \setminus A$. Prove that, when A and B are subsets of X and $\mathcal{C}$ is a set of subsets of X, that (these are known as de Morgan's laws)

$$A^{\mathsf{c}} \cap B^{\mathsf{c}} = (A \cup B)^{\mathsf{c}}, \qquad \bigcap\{C^{\mathsf{c}} \colon C \in \mathcal{C}\} = \Bigl(\bigcup \mathcal{C}\Bigr)^{\mathsf{c}},$$

$$A^{\mathsf{c}} \cup B^{\mathsf{c}} = (A \cap B)^{\mathsf{c}}, \qquad \bigcup\{C^{\mathsf{c}} \colon C \in \mathcal{C}\} = \Bigl(\bigcap \mathcal{C}\Bigr)^{\mathsf{c}}.$$

Also, when $f \colon Y \to X$ then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \qquad f^{-1}\Bigl(\bigcap C\Bigr) = \bigcap\{f^{-1}(C) \colon C \in \mathcal{C}\},$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \qquad f^{-1}\Bigl(\bigcup C\Bigr) = \bigcup\{f^{-1}(C) \colon C \in \mathcal{C}\},$$

$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B).$$

Exercise A.2. Let X and Y be topological spaces. Show that a function $f \colon X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X for every closed $F \subseteq Y$.

Exercise A.3. We have defined compactness of X in terms of open covers of X, which are sets of open subsets of X covering X.

Instead, consider now a subset K of a topological space X. Then K with the topology inherited from X is a topological space in its own right, so we can ask if K it compact or not.

If we define an open cover of K to be a set of open subsets of X whose union contains K, prove that K is compact if and only if every open cover of K has a finite subcover (of K). (The point here is that compactness of K is defined in terms of open covers of K, as consisting of subsets of K which are open in the inherited topology.)

Exercise A.4. Show that any closed subset of a compact space is compact.

Exercise A.5. Show that any compact subset of a Hausdorff space is closed.

Exercise A.6. Let X be a compact Hausdorff space.

(a) Show that if you replace the topology on X by a strictly stronger topology, then in the new topology X is still Hausdorff but no longer compact.

(b) Show that if you replace the topology on X by a strictly weaker topology, then in the new topology X is still compact but no longer Hausdorff.

Hint. Show both the easy parts first (Hausdorff in (a), compact in (b)). For (a) and (b) both, assume F is closed in the stronger topology but not in the weaker topology, and consider the compactness (or not) of F in the two topologies to arrive at a contradiction.

Exercise A.7. Let X be a set and $\mathcal{B}$ a set of subsets of X. Let $\mathcal{B}'$ be the set of all finite intersections from $\mathcal{B}$:

$$\mathcal{B}' = \{B_1 \cap \cdots \cap B_n \colon B_1, \ldots, B_n \in \mathcal{B}; n = 0, 1, 2, \ldots\}$$

with the understanding that $B_1 \cap \cdots \cap B_n = X$ when $n = 0$. Let $\mathcal{T}$ consist of all possible unions of members of $\mathcal{B}'$. Show that $\mathcal{T}$ is a topology; in fact, it is the weakest topology containing $\mathcal{B}$. It is said to be the topology *generated* by $\mathcal{B}$. Also, $\mathcal{B}$ is said to be a *basis* for T.

Exercise A.8. A topology is called *second countable* if it has a countable basis. It is called *first countable* if every neighbourhood filter has a countable filter base. Show that any metrizable space is first countable. Also show that any second countable space is first countable. Give an example of a first countable space that is not second countable.