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For questions during the exam contact:
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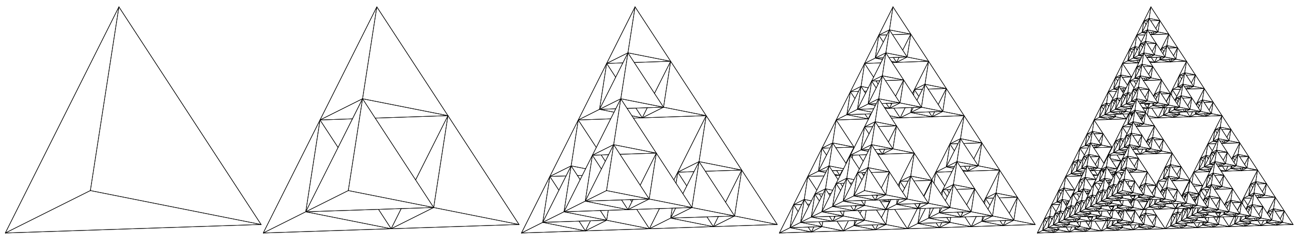
EXAM IN 75045 DYNAMICAL SYSTEMS

Tuesday 19 mai 1998
Time: 09:00–15:00

Aids: B1
— Approved calculator, with empty memory
— No printed or written aids allowed

Reasons must be given for all answers

Problem 1 The midpoints of the six edges of a tetrahedron are the corners of an octahedron. If this octahedron is removed from the tetrahedron, one is left with four smaller tetrahedra. Repeat this procedure with each small tetrahedron, and so forth, so that after n rounds we are left with 4^n tetrahedra, each with edges of length 2^{-n} times the lengths of the original tetrahedron. What is the fractal dimension of the set ("Sierpinski's tetrahedron") which appears in the limit as $n \to \infty$?



A larger version of the result after the fifth iteration appears on the last page.

Problem 2 (Counts as two questions)

Briefly discuss the various kinds of equilibrium points which may appear in a two-dimensional linear dynamical system $\dot{x} = Ax$ where A is a non-singular 2×2 matrix, and sketch the phase portraits around these.

Problem 3

a) Determine the equilibrium points and their linear and nonlinear type for the system

$$\dot{x} = x^2 - y^2,$$
$$\dot{y} = 1 - x^2 - y^2.$$

Is there a periodic trajectory surrounding all the equilibrium points?

b) Sketch a phase diagram for the system.

Problem 4 Investigate whether the origin is an asymptotically stable, stable, or unstable equilibrium point for the system

$$\dot{x} = 2y - x^3,$$
$$\dot{y} = -4x^3.$$

Problem 5 Investigate whether the system

$$\dot{x} = y,$$
$$\dot{y} = -x - y - y^3 + \frac{2y}{1+x^2}$$

has any periodic trajectories.

Problem 6 Assume given a system of differential equations on the form

$$\dot{x} = g(x,y) - y^3,$$
$$\dot{y} = h(x,y) - x^3$$

where g and h are C^1 functions with the property that

$$|g(x,y)| \leq A(1 + x^2 + y^2), \quad |h(x,y)| \leq A(1 + x^2 + y^2)$$

for a constant A. Show that the system has at least one equilibrium. What can you say about this point, if you are told there is only one equilibrium, and that it is nondegenerate?

Problem 7

a) Assume given a Lipschitz continuous vector field f on an open set $E \subset \mathbb{R}^n$ and a point $x_0 \in E$. Show that the initial value problem

$$\dot{x}(t) = f\big(x(t)\big), \quad x(0) = x_0$$

has at most one solution for t in an interval $[0, T]$ with $T > 0$. Give an example showing that this may not hold if we drop the condition of Lipschitz continuity.

b) What is meant by the the flow of the vector field f? What are its fundamental properties? Give an example showing that the flow need not be defined everywhere even if $E = \mathbb{R}^n$.