MA2104 Fall 2006, Week 46: Solutions to exercises

Problem MNFMA213 1998–12–10 #5:

a) The equation becomes $4X''T = XT''$. Standard methods yield $X = \sin nx$, $n = 1, 2, 3, \ldots$ so the T equation becomes $T'' = -4n^2T$. The answer is on the form

$$u_n(x,t) = \sin nx \cdot (b_n \cos 2nt + b_n^* \sin 2nt), \qquad n = 1, 2, 3, \ldots$$

b) Adding up solutions as above, we have a more general candidate for a solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \cdot (b_n \cos 2nt + b_n^* \sin 2nt).$$

The given initial data become

$$\sum_{n=1}^{\infty} b_n \sin nx = \sin 2x + 3 \sin 5x, \quad \sum_{n=1}^{\infty} 2nb_n^* \sin nx = 3 \sin 4x - \sin 3x.$$

These are trivial to satisfy – the righthand sides *are already* tiny little Fourier series! So we just match coefficients: $b_2 = 1$, $b_5 = 3$, all other $b_n = 0$, and $8b_4^* = 3$, $6b_3^* = -1$, all other $b_n^* = 0$.

$$u(x,t) = \sin 2x \cos 4t - \tfrac{1}{6} \sin 3x \sin 6t + \tfrac{3}{8} \sin 4x \sin 8t + 3 \sin 5x \cos 10t.$$

Problem MNFMA214 2002–05–16 #2:

a) The given function is rational, so all its singularities are poles. We factor the denominator:

$$z^3 - 3z^2 - 2z = (z^2 - 3z - 2)z = \bigl((z - \tfrac{3}{2})^2 - \tfrac{17}{4}\bigr)z = (z - \tfrac{3}{2} - \tfrac{1}{2}\sqrt{17})(z - \tfrac{3}{2} + \tfrac{1}{2}\sqrt{17})z$$

(it is of course OK to use the formula for the solution of the quadratic equation to find the roots).

All of the zeroes $z = z_\pm = \frac{3}{2} \pm \frac{1}{2}\sqrt{17}$, $z = 0$ of the denominator are simple, and they are not zeroes of the numerator, so each is a simple pole.

b) *Note*: *Unless I have made a mistake, this question produces way too much calculation for a good exam problem. There is a good chance that I did make a mistake in there somewhere, so read this solution with a critical eye. And let me know if you find a mistake.* (*I don't quite have time to check the solution carefully.*)

Apart from $z = 0$, the pole closest to the origin is $z = \frac{3}{2} - \frac{1}{2}\sqrt{17} \approx \frac{3}{2} - \frac{4}{2} = -\frac{1}{2}$, so the problem is wrong in assuming there is a Laurent series valid for $0 < |z| < 1$. We shall have to settle for a series valid for $0 < |z| < \frac{1}{2}\sqrt{17} - \frac{3}{2}$ instead.

I think the easiest way to get such a series is to perform a partial fraction decomposition of the function, and for that it is handy to have all the residues (this technique only works when all the poles are simple):

$$\operatorname{Res}(f, 0) = \left.\frac{3z^2 - 6z + 2}{z^2 - 3z - 2}\right|_{z=0} = -1,$$

and

$$\operatorname{Res}(f, z_\pm) = \left.\frac{3z^2 - 6z + 2}{(z - z_\mp)z}\right|_{z=z_\pm} = \frac{3z_\pm^2 - 6z_\pm + 2}{(z_\pm - z_\mp)z_\pm}$$
$$= \frac{3z_\pm - 6 + 2/z_\pm}{\pm\sqrt{17}} = \frac{3z_\pm - 6 - z_\mp}{\pm\sqrt{17}} = \frac{-3 \pm 2\sqrt{17}}{\pm\sqrt{17}} = 34 \mp 3\sqrt{17}$$

In the second row, I have used the fact that $z_+ z_- = -2$ (since they are the zeroes of $z^2 - 3z - 2$: the constant term is the product of the roots).

Now, the whole point of this is that in the partial fraction decomposition

$$f(z) = \frac{A}{z} + \frac{B_+}{z - z_+} + \frac{B_-}{z - z_-}$$

we can immediately read off the residues: They are A at $z = 0$ and $B_\pm$ at $z = z_\pm$. Turning this around, since we know the residues that means

$$A = -1, \quad B_\pm = \operatorname{Res}(f, z_\pm) = 34 \mp 3\sqrt{17},$$

so that

$$f(z) = -\frac{1}{z} + \frac{34 - 3\sqrt{17}}{z - z_+} + \frac{34 + 3\sqrt{17}}{z - z_-}$$

and the desired Laurent series is

$$\begin{aligned} f(z) &= -\frac{1}{z} - (34 - 3\sqrt{17}) \sum_{n=0}^{\infty} \frac{z^n}{z_+^{n+1}} - (34 + 3\sqrt{17}) \sum_{n=0}^{\infty} \frac{z^n}{z_-^{n+1}} \\ &= -\frac{1}{z} - \sum_{n=0}^{\infty} \left(\frac{34 - 3\sqrt{17}}{(\frac{3}{2} + \frac{1}{2}\sqrt{17})^{n+1}} + \frac{34 - 3\sqrt{17}34 + 3\sqrt{17}}{(\frac{3}{2} - \frac{1}{2}\sqrt{17})^{n+1}} \right) z^n. \end{aligned}$$

c) Fortunately, we do not need the full results from above to answer this one. We only need to know that only one pole, the one at $z = 0$, is inside the given circle. (The next one is *just* outside it.) And the residue at $z = 0$ was the easiest one to compute above: It is -1. So the integral is $-2\pi i$.

Problem MNFMA214 2003–05–19 #5: The function $e^{f(z)}$ is also entire, and $|e^{f(z)}| = e^{\operatorname{Re} f(x)} \le e^M$. By Liouville's theorem, $e^{f(z)}$ is constant. Hence so is f.

Problem MA2104 2004–12–13 #3:

a) The singular points are where $\cosh z = 0$. That is $e^z + e^{-z} = 0$, or multiplying by e^z we get $e^{2z} + 1 = 0$. This happens precisely when $2z = (2k+1)i\pi$ for $k \in \mathbb{Z}$, in other words when $z = (k + \frac{1}{2})i\pi$, $k \in \mathbb{Z}$.

The derivative of $\cosh z$ at these points is $\sinh z = \sinh(k + \frac{1}{2})i\pi = i \sin(k + \frac{1}{2})\pi = (-1)^k \ne 0$, so the poles are simple.

b) The rectangle is a closed contour surrounding precisely one of the poles, namely the one at $z = \frac{1}{2}i\pi$. The residual there is $1/\sinh \frac{1}{2}i\pi = 1/i$, so the given integral is $2\pi i/i = 2\pi$.

c) We need to show that $\cosh z$ becomes large: Write $z = x + iy$ with $x = \pm R$ and $0 \le y \le \pi$.

Now when $x = R$ then $|e^z| = e^x = e^R$ and $|e^{-z}| = e^{-x} = e^{-R} < 1$, so $|\cosh z| = \frac{1}{2}|e^z + e^{-z}| \ge \frac{1}{2}e^R - 1$. One gets the same estimate when $x = -R$. So $|f(z)| \le 2/(e^R - 1)$ for z on one of the vertial sides of the rectangle. Therefore the absolute value of the integral along one of the vertical sides is at most $2\pi/(e^R - 1) \to 0$ as $R \to \infty$.

d) We find $\cosh(x + \pi i) = -\cosh x$, so the integrals along the top and bottom of the rectangle are equal when performed with the orientation shown. So $\int_{-\infty}^{\infty} f(x)\,dx = \pi$. By symmetry, the requested integral is half that:

$$\int_0^\infty \frac{1}{\cosh x}\,dx = \frac{\pi}{2}.$$

Problem MA2104 2004–12–13 #5: The given integral is a path integral:

$$\int_0^{2\pi} \frac{\cos\theta}{\sqrt{2}+\cos\theta}\,d\theta = \int_{C_1(0)} \frac{\frac{1}{2}(z+z^{-1})}{\sqrt{2}+\frac{1}{2}(z+z^{-1})}\frac{dz}{iz} = -i\int_{C_1(0)} \frac{z^2+1}{(z^2+2\sqrt{2}z+1)z}\,dz$$

We need to further factor the denominator: $z^2+2\sqrt{2}z+1 = (z+\sqrt{2})^2-1 = (z+1+\sqrt{2})(z-1+\sqrt{2})$. Only one zero is within the unit circle, namely $z = 1-\sqrt{2}$.

The integrand has a pole at $z = 0$, with residue 1, and another pole at $z = 1-\sqrt{2}$, with residue

$$\frac{z^2+1}{(z+1+\sqrt{2})z}\bigg|_{z=1-\sqrt{2}} = \frac{4-2\sqrt{2}}{2(1-\sqrt{2})} = -\sqrt{2}.$$