MA2104 Fall 2006, Week 45: Solutions to exercises

Problem 5.1.15: The function has a triple pole (pole of order 3) at $z = 1 + i$ and a single pole at $z = i$. The latter is outside the given contour, so we only need worry about the triple one. Perhaps the easiest recipe for the residue in this case is (8) on p. 322 in the book, which becomes

$$\operatorname{Res}\Bigl(\frac{z+i}{(z-1-i)^3(z-i)}\Bigr) = \lim_{z\to 1+i} \frac{1}{2!}\Bigl(\frac{d}{dz}\Bigr)^2 \frac{z+i}{z-i}$$
$$= \frac{1}{2!}\Bigl(\frac{d}{dz}\Bigr)^2 \Bigl(1 + \frac{2i}{z-i}\Bigr)\Bigg|_{z=1+i} = \frac{2i}{(z-i)^3}\Bigg|_{z=1+i} = 2i.$$

(In the second equality I used $z + i = z - i + 2i$ to simplify the calculation.) So the integral is $2\pi i \cdot 2i = -4\pi$.

Problem 5.1.20: The integrand has a singularity at any point where $e^{\pi z} = -1$. That is, where $\pi z = (2k+1)\pi i$, for $k \in \mathbb{Z}$, The only such point within the integration contour is $z = -i$.

Since the derivative of the denominator is $\pi e^{\pi z} \neq 0$ everywhere, each zero of the denominator is a simple zero, and so the poles of the integrand are all simple. Moreover,

$$\operatorname{Res}\Bigl(\frac{1}{1+e^{\pi z}}, -i\Bigr) = \frac{1}{d(1+e^{\pi z})/dz}\Bigg|_{z=-i} = \frac{1}{\pi e^{\pi z}}\Bigg|_{z=-i} = -\frac{1}{\pi}.$$

So the integral is $-2\pi i/\pi = -2i$.

Problem 5.1.25: The integrand has a quintuple pole at $z = 0$ (the denominator has a zero of order 6, but the simple zero in the numerator lowers the order of the pole to 5).

We can differentiate $\sin z$ five times, put $z = 0$ and divide by 5! to get the residue $(\cos 0)/5! = 1/5!$, so the integral is $2\pi i/5!$.

Alternatively, we can use the known power series for $\sin z$:

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k-5}$$

The z^{-1} term in this series is the one with $k = 2$, and the corresponding coefficient gives the residue at zero: Its value is $(-1)^2/5! = 1/5!$ as before.

Problem 5.2.7: The half-angle trick is useful to keep the degree of polynomials down. Just remember the formula $\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$. The final one is useful here: It gives $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. Insert that into the given integral and change the variable (put $2\theta = \varphi$, then rename φ back to θ):

$$I = \int_0^{\pi} \frac{d\theta}{9 + 16\sin^2\theta} = \int_0^{\pi} \frac{d\theta}{17 - 8\cos 2\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta}.$$

Next, we parametrize the unit circle using $z = e^{i\theta}$, so that $dz = ie^{i\theta}\,d\theta$, or $d\theta = -i\,dz/z$:

$$I = -\frac{i}{2}\int_{C_1(0)} \frac{dz}{z(17 - 4z - 4z^{-1})} = \frac{i}{8}\int_{C_1(0)} \frac{dz}{z^2 - \frac{17}{4}z + 1}.$$

We factor the denominator of the integral as follows:

$$z^2 - \tfrac{17}{4}z + 1 = \bigl(z - \tfrac{17}{8}\bigr)^2 - \tfrac{225}{64} = \bigl(z - \tfrac{17}{8} - \tfrac{15}{8}\bigr)\bigl(z - \tfrac{17}{8} + \tfrac{15}{8}\bigr) = (z-4)\bigl(z - \tfrac{1}{4}\bigr)$$

(In the process of this calculation, I became aware of the Pythagorean triple $(8, 15, 17)$, which I had forgotten about.)

Only the zero $z = \frac{1}{4}$ is inside the unit circle, and the residue there will be

$$\left.\frac{1}{z-4}\right|_{z=1/4} = -\frac{4}{15},$$

so the integral is (don't forget the factor $i/8$ that we put outside)

$$2\pi i \cdot \frac{i}{8} \cdot \frac{-4}{15} = \frac{\pi}{15}.$$

Problem 5.2.11: Same as the previous problem, except we don't need the half-angle trick:

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = -i \int_{C_1(0)} \frac{dz}{z(1 + \frac{1}{2}az + \frac{1}{2}az^{-1})} = -\frac{2i}{a} \int_{C_1(0)} \frac{dz}{z^2 + \dfrac{2z}{a} + 1}$$

Again, factor the denominator of the integrand:

$$\begin{aligned} z^2 + \frac{2z}{a} + 1 &= \left(z + \frac{1}{a}\right)^2 + 1 - \frac{1}{a^2} = \left(z + \frac{1}{a}\right)^2 - \frac{1-a^2}{a^2} \\ &= \left(z + \frac{1 - \sqrt{1-a^2}}{a}\right)\left(z + \frac{1 + \sqrt{1-a^2}}{a}\right). \end{aligned}$$

The product of the two zeros

$$z_\pm = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

is one, and clearly $|z_-| > 1/|a| > 1$, so $|z_+| < 1$, and z_+ is the one pole inside the unit circle. The residue at that pole will be

$$\frac{1}{z - z_-}\bigg| z = z_+ = \frac{1}{z_+ - z_-} = \frac{a}{2\sqrt{1-a^2}}.$$

Don't forget the factor $-2i/a$ that we left outside the integral. The final answer for the integral will be

$$2\pi i \cdot \frac{-2i}{a} \cdot \frac{a}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

Problem 5.2.13: The double angle trick yields $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, so we substitute that, replace 2θ by θ, and finally note that we are integrating over two periods, so we integrate over one period instead and double the answer:

$$\begin{aligned} I = \int_0^{2\pi} \frac{d\theta}{a + b\cos^2\theta} &= \int_0^{2\pi} \frac{d\theta}{a + \frac{1}{2}b + \frac{1}{2}b\cos 2\theta} = \frac{1}{2}\int_0^{4\pi} \frac{d\theta}{a + \frac{1}{2}b + \frac{1}{2}b\cos\theta} \\ &= \int_0^{2\pi} \frac{d\theta}{a + \frac{1}{2}b + \frac{1}{2}b\cos\theta}. \end{aligned}$$

Next we could rewrite this as a contour integral as before, but it is a lot easier to just rewrite this integral a bit further, so it looks like the integral we solved in the previous problem! In fact

$$I = \frac{1}{a + \frac{1}{2}b}\int_0^{2\pi} \frac{d\theta}{1 + A\cos\theta}, \qquad A = \frac{b}{2a+b}$$

so that

$$I = \frac{2\pi}{(a + \frac{1}{2}b)\sqrt{1-A^2}} = \frac{2\pi}{\sqrt{a(a+b)}}.$$

For this to be valid we need $|A| < 1$. But given that a and b are positive, this is equivalent to the given condition $b < a$.

Problem 8.2.8: The equation with $c = 1/\pi$ is

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}$$

The usual separation of variables with $u = XT$ yields $XT'' = X''T/\pi^2$, which becomes $X''/X = \pi^2 T''/T$. Both sides have to be constants.

Now the length of the string is 1 and endpoints are held fixed, which yield boundary conditions $X(0) = X(1) = 1$. Thus the only nontrivial solutions are of the form (constant times) $X = \cos n\pi x$ with $n = 1, 2, \ldots$.

For the T equations we find $T'' = -n^2 T$, with solutions $T = b_n \cos nt + b_n^* \sin nt$. So we have the candidate solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \cdot (b_n \cos nt + b_n^* \sin nt).$$

The intial conditions $u(x,0) = f(x)$ and $\partial u/\partial t(x,0) = g(x)$ become

$$\sum_{n=1}^{\infty} b_n \sin n\pi x = f(x), \qquad \sum_{n=1}^{\infty} n b_n^* \sin n\pi x = g(x).$$

Since $g(x) = 0$, $b_n^* = 0$, the coefficients b_n must be the Fourier sine coefficients of f:

$$\begin{aligned} b_n &= 2\int_0^1 f(x) \sin n\pi x \, dx = 2\int_0^1 x \sin \pi x \sin n\pi x \, dx \\ &= \int_0^1 x\big(\cos(n-1)\pi x - \cos(n+1)\pi x\big)\, dx. \end{aligned}$$

A quick partial integration gives

$$\int_0^1 x \cos m\pi x \, dx = \frac{1}{m\pi}\Big[x \sin m\pi x\Big]_{x=0}^{1} - \frac{1}{m\pi}\int_0^1 \sin m\pi x \, dx = \frac{(-1)^m - 1}{m^2\pi^2}$$

for $m = 1, 2, 3, \ldots$ while for $m = 0$ the answer is clearly $\frac{1}{2}$. We plug this into the formula for b_n and get $b_1 = 1$, $b_n = 0$ for $n = 3, 5, 7, \ldots$ and

$$b_n = -\frac{2}{\pi^2}\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right) = -\frac{8n}{\pi^2(n^2-1)^2}, \qquad n = 2, 4, 6, \ldots$$

so that

$$u(x,t) = \sin \pi x \cdot \sin t - \frac{8}{\pi^2}\sum^{\infty} \frac{2k}{\big((2k)^2 - 1\big)^2} \sin n\pi x \cdot \sin nt$$

Problem 2003–12–15 #1:
The solution for this entire exam is posted on the web (in Norwegian.)

Problem 2003–12–15 #4:
The solution for this entire exam is posted on the web (in Norwegian.)