MA2104 Fall 2006, Week 44: Solutions to exercises

Problem 4.4.22: The partial fraction decomposition of the given expression is

$$\frac{1}{(z_1-z)(z_2-z)} = \frac{1}{z_1-z_2}\Big(\frac{1}{z_2-z} - \frac{1}{z_1-z}\Big)$$

now when $|z| < |z_1|$ we can write

$$\frac{1}{z_1-z} = \frac{1}{z_1}\frac{1}{1-z/z_1} = \frac{1}{z_1}\sum_{n=0}^{\infty}\Big(\frac{z}{z_1}\Big)^n = \sum_{n=0}^{\infty}\frac{z^n}{z_1^{n+1}}$$

and similarly

$$\frac{1}{z_2-z} = \sum_{n=0}^{\infty}\frac{z^n}{z_2^{n+1}}$$

obviously, and together with the above partial fraction decomposition we conclude

$$\frac{1}{(z_1-z)(z_2-z)} = \frac{1}{z_1-z_2}\sum_{n=0}^{\infty}\Big(\frac{1}{z_2^{n+1}} - \frac{1}{z_1^{n+1}}\Big)z^n$$

– I have no idea why the book uses the less simple form that it does.

Problem 4.4.23: Look for the zeros of the denominator, either by using the formula for solutions to the quadratic equation or (my preference) completing the square:

$$1+z+z^2 = \big(z+\tfrac12\big)^2 + \tfrac34 = \big(z+\tfrac12+\tfrac{i}{2}\sqrt3\big)\big(z+\tfrac12-\tfrac{i}{2}\sqrt3\big) = (z_- - z)(z_+ - z)$$

where

$$z_\pm = -\tfrac12 \pm \tfrac{i}{2}\sqrt3 = e^{\pm 2\pi i/3}.$$

So we get $z_\pm^n = e^{\pm 2n\pi i/3}$ for the two singularities of the function.[1] Since these have absolute value 1, that means the largest disk centered at 0 in which the function is analytic has radius 1. And so the radius of convergence of the Maclaurin series will be 1: This we can conclude without even computing it.

Now

$$z_\pm^m = \begin{cases} 1 & m = 3k, \\ z_\pm & m = 3k+1, \\ z_\mp & m = 3k-1, \end{cases}$$

(where k is any integer), so that

$$z_+^m - z_-^m = \begin{cases} 0 & m = 3k, \\ i\sqrt3 & m = 3k+1, \\ -i\sqrt3 & m = 3k-1. \end{cases}$$

We shall need this for the case $m = -n-1$ in order to use the result from the previous

[1] This looks familiar, and with hindsight we could have shortened the computation by noting that $(1-z)(1+z+z^2) = 1-z^3$, which has three zeros of the form $z = e^{2\pi ik/3}$, of which the one with $k=0$ (i.e., $z=1$) came from the factor $1-z$, while the other two must come from $1+z+z^2$.

exercise:

$$
\begin{aligned}
\frac{1}{1+z+z^2} &= \frac{1}{z_- - z_+}\sum_{n=0}^{\infty}\Bigl(\frac{1}{z_+^{n+1}} - \frac{1}{z_-^{n+1}}\Bigr)z^n \\
&= \frac{i}{\sqrt{3}}\Biggl(\sum_{k=0}^{\infty}\bigl(z_+^{-3k-1} - z_-^{-3k-1}\bigr)z^{3k} \\
&\qquad + \sum_{k=0}^{\infty}\bigl(z_+^{-3k-2} - z_-^{-3k-2}\bigr)z^{3k+1} + \sum_{k=0}^{\infty}\bigl(z_+^{-3k-3} - z_-^{-3k-3}\bigr)z^{3k+2}\Biggr) \\
&= \frac{i}{\sqrt{3}}\Bigl(-i\sqrt{3}\sum_{k=0}^{\infty} z^{3k} + i\sqrt{3}\sum_{k=0}^{\infty} z^{3k+1}\Bigr) \\
&= \sum_{k=0}^{\infty} z^{3k} - \sum_{k=0}^{\infty} z^{3k+1}
\end{aligned}
$$

which looks almost too simple to be true. But we can easily add these two geometric series to check the result:

$$
\sum_{k=0}^{\infty} z^{3k} - \sum_{k=0}^{\infty} z^{3k+1} = \frac{1}{1-z^3} - \frac{z}{1-z^3} = \frac{1-z}{1-z^3} = \frac{1}{1+z+z^2}
$$

where I used the formula noted in a footnote. Of course, if we were smarter we would use this formula at the beginning and get out of a big calculation.[2]

Problem 4.5.1: For $|z| > 1$:

$$
\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+z^{-2}} = \frac{1}{z^2}\sum_{k=0}^{\infty} z^{-2k} = \sum_{k=0}^{\infty} z^{-2k-2} = \sum_{k=1}^{\infty} z^{-2k}.
$$

Problem 4.5.13: Note the partial fraction decomposition (I removed the factor z for simplicity; better remember to put it back in later):

$$
\frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3}.
$$

Since $2 < |z| < 3$ we need to handle the two fractions differently, in each case setting the larger summand outside (I'll do them simultaneously):

$$
\begin{aligned}
\frac{1}{z+2} - \frac{1}{z+3} &= \frac{1}{z}\frac{1}{1+2/z} - \frac{1}{3}\frac{1}{1+z/3} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\Bigl(\frac{2}{z}\Bigr)^n - \frac{1}{3}\sum_{n=0}^{\infty}\Bigl(\frac{z}{3}\Bigr)^n = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}
\end{aligned}
$$

where the first sum is the singular part, and the second the regular part, of the Laurent series.

Problem 4.5.25: We just substitute $1/z$ for z in the usual Taylor series for $\sin z$ to get

$$
\sin\frac{1}{z} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k+1}}
$$

The term containing $1/z$ is the one with $k = 0$, and indeed has the simple form $1/z$. So the integral around $C_1(0)$ is $2\pi i$.

[2] There is perhaps a lesson buried here: Sometimes people discover simple relations like this after going through a lengthy calculation and getting a surprisingly simple result, and then they present only the quick way to get there. The result is that it looks like a piece of magic – how did the person think of this in the first place? So whenever you see this happening, you may as well assume that most likely, the discoverer of the quick way got there in the obvious but hard way first time around.

Problem 7.5.15: A Taylor expansion valid for $|z| < 2$ can be obtained from a simple geometric series:

$$\frac{z}{2+z^2} = \frac{z}{2}\frac{1}{1+z^2/2} = \frac{z}{2}\sum_{k=0}^{\infty}\Bigl(-\frac{z^2}{2}\Bigr)^k = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{2^{k+1}}.$$

Now substitute $z = e^{i\theta}$ to get the desired Fourier series on the form

$$\frac{e^{i\theta}}{2+e^{2i\theta}} = \sum_{k=0}^{\infty}\frac{e^{(2k+1)i\theta}}{2^{k+1}}.$$

Problem 7.5.17: Same trick: Substitute $z = e^{i\theta}$ in the series $e^z = \sum_{n=0}^{\infty} z^n/n!$ to get

$$e^{e^{i\theta}} = \sum_{n=0}^{\infty}\frac{e^{in\theta}}{n!}$$

Problem 8.2.5(a): We just use the formulas on page 520, with $L = 1$ and $g = 0$: From formula (9) on that page $b_n^* = 0$ and

$$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx = \tfrac{1}{30}\int_{1/3}^{2/3}(x-\tfrac{1}{3})\sin n\pi x\,dx + \tfrac{1}{30}\int_{2/3}^{1}(1-x)\sin n\pi x\,dx$$

I admit I used Maple to solve these integrals. They are not hard (it's just partial integration), but it's a bit of work to do it right. Maple's answer is

$$b_n = \frac{2\sin\frac{2}{3}\pi n - \sin\frac{1}{3}\pi n}{30\pi^2 n^2}.$$

We *could* use

$$\sin\tfrac{1}{3}\pi n = \begin{cases} 0 & n = 6k \text{ or } n = 6k+3, \\ \frac{1}{2}\sqrt{3} & n = 6k+1 \text{ or } n = 6k+2, \\ -\frac{1}{2}\sqrt{3} & n = 6k+4 \text{ or } n = 6k+5, \end{cases}$$

but that seems like a lot of work to write it up in a meaningful way, so I'll stop here, and just record the answer:

$$u(x,t) = \sum_{n=1}^{\infty}\sin n\pi x \cdot b_n \cos nt$$

where b_n is given above.

Problem 8.2.9(a): The Fourier sine series of $f(x) = x(1-x)$ on $[0,1]$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ where

$$b_n = 2\int_0^1 x(1-x)\sin n\pi x\,dx = 4\frac{1-(-1)^n}{\pi^3 n^3} = \begin{cases} 0 & n \text{ even}, \\ \dfrac{8}{\pi^3 n^3} & n \text{ odd}. \end{cases}$$

Now g is also supposed to be expanded as a sine series, but $g(x) = \sin \pi x$ is *already* written as a sine series, so there is no more work to do. If you insist, think of it as $g(x) = \sum_{n=1}^{\infty} b_n^* \sin n\pi x$ where $b_1^* = 1$ and all the other b_n^* are zero.

Again, all we have to do is record the final answer as

$$u(x,t) = \sin\pi x \sin\pi t + \sum_{k=0}^{\infty}\frac{8}{\pi^3(2k+1)^3}\sin(2k+1)\pi x \cdot \sin(2k+1)\pi t$$

where I put the term arising from g first to avoid confusion. Note that this term might be combined with the $k = 0$ term of the sum.