MA2104 Fall 2006, Week 43: Solutions to exercises

Since the function e^{-z^2} is entire, its integral around any closed path is zero according to Cauchy's theorem.

For the given triangular path $[0, M, M+iM, 0]$ we may wish to consider the integral along the hypothenuse in reverse, so we write this as

$$\underbrace{\int_{[0,M]} e^{-z^2}\,dz}_{A(M)} + \underbrace{\int_{[M,M+iM]} e^{-z^2}\,dz}_{B(M)} - \underbrace{\int_{[0,M+iM]} e^{-z^2}\,dz}_{C(M)} = 0. \tag{1}$$

Now

$$A(M) = \int_0^M e^{-x^2}\,dx \to \frac{\sqrt{\pi}}{2} \qquad \text{as } M \to +\infty. \tag{2}$$

Next, with the parametrization of $M, M+iM$ given by $z = M+iy$ with $0 \le y \le M$ we find

$$B(M) = i\int_0^M e^{-(M+iy)^2}\,dy = i\int_0^M e^{-M^2+y^2+2iMy}\,dy$$

so that

$$\bigl|B(M)\bigr| \le \int_0^M |e^{-M^2+y^2+2iMy}|\,dy = \int_0^M e^{-M^2+y^2}\,dy = \int_0^M e^{(y+M)(y-M)}\,dy.$$

Now in the integration interval $y-M \le 0$ and $y+M \ge M$, so $(y+M)(y-M) \le M(y-M)$ and therefore

$$\bigl|B(M)\bigr| \le \int_0^M e^{M(y-M)}\,dy = \frac{1-e^{-M^2}}{M} < \frac{1}{M}$$

so that

$$B(M) \to 0 \qquad \text{as } M \to +\infty. \tag{3}$$

Combining equations (1), (2) and (3), we conclude that

$$C(M) \to \frac{\sqrt{\pi}}{2} \qquad \text{as } M \to +\infty. \tag{4}$$

Finally, we integrate along the hypothenuse $[0, M+iM]$ by setting $z = (x+ix)/\sqrt{2}$ for $0 \le x \le \sqrt{2}M$, so that $dz = (1+i)/\sqrt{2}dx$, and also $z^2 = -x^2$ (which is my reason for dividing by $\sqrt{2}$ – it saves a change of variables later)

$$C(M) = \frac{1+i}{\sqrt{2}}\int_0^M e^{-ix^2}\,dx = \frac{1}{\sqrt{2}}\int_0^M (\cos x^2 + \sin x^2)\,dx + \frac{i}{\sqrt{2}}\int_0^M (\cos x^2 - \sin x^2)\,dx.$$

But equation (4) tells us this has a real limit as $M \to \infty$, so the imaginary part of the above – that is the final integral – must go to zero, while the real part – the next-to-last integral – must converge to the righthand side of (4). From the first of these two conclusions we find

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx,$$

and substituting this into the second conclusion we find

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\int_0^\infty (\cos x^2 + \sin x^2)\,dx = \sqrt{\frac{\pi}{2}}.$$

(The finite versions of these integrals are known as *Fresnel integrals*. Together, they parametrize the so-called *Cornu spiral*.)