MA2104 Fall 2006, Week 42: Solutions to exercises

Some pictures are at the end.

Problem 7.1.8: (a) We are given $f(x) = \cos x + \cos \pi x$ and asked to show that there is only one solution to $f(x) = 2$. (Clearly, what is meant is that there is only one *real* solution. I am sure there are many complex solutions, though I have not worked out the details.) Since $\cos x \le 1$ and $\cos \pi x \le 1$ as well, their sum is always ≤ 2, with equality if and only if both summands equal 1. From $\cos x = 1$ we get $x = m\pi$ with $m \in \mathbb{Z}$, and from $\cos \pi x = 1$ we get $\pi x = n\pi$ with $n \in \mathbb{Z}$. Dividing the two formulas, we get $\pi = n/m$, which is a contradiction because π is irrational. The division is not allowed in the special case $x = 0$, which obviously is a solution and therefore the only solution.

(b) If f were periodic with period T, then we would have $f(nT) = 2$ for all $n \in \mathbb{Z}$ because $f(0) = 2$.

Problem 7.1.12: Since the integral is over a whole period, we can move the interval wherever we wish, so

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^\pi f(x)\,dx = \int_0^\pi \cos x\,dx = \sin\pi - \sin 0 = 0.$$

Problem 7.1.15: If f and $F(x) = \int_a^x dx$ are 2π-periodic, then $F(a + 2\pi) = F(a) = 0$, so $\int_a^{a+2\pi} f(x)\,dx = 0$. Then $\int_0^{2\pi} f(x)\,dx = 0$ as well (by Theorem 1).

On the other hand, if $\int_0^{2\pi} f(x)\,dx = 0$ then (once more using Theorem 1)

$$F(x + 2\pi) - F(x) = \int_x^{x+2\pi} f(x), dx = \int_0^{2\pi} f(x), dx = 0,$$

so F is 2π-periodic.

In general, if $\int_0^{2\pi} f(x)\,dx = b$ then $F(x + 2n\pi) = F(x) + nb$ whenever $n \in \mathbb{Z}$, so F grows roughly linearly. More precisely, $F(x)$ is the sum of bx and a 2π-periodic function.

Problem 7.2.5: (a) Since the given function is even, its Fourier series will be a cosine series. On the interval $[0, \pi]$ we have $f(x) = x$, which simplifies the integral somewhat. For the coefficients we find

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi n}\Big[x\sin nx\Big]_0^\pi - \frac{2}{\pi n}\int_0^\pi \sin nx\,dx = \frac{2}{\pi n^2}\big((-1)^n - 1\big)$$

which is zero for even n, leaving the nonzero coefficients as

$$a_{2k+1} = \frac{4}{\pi(2k+1)^2}, \qquad k = 0, 1, 2, \ldots$$

resulting in the Fourier series given in the problem.

(b) The function is continuous and piecewise smooth, so its Fourier series converges uniformly to the function everywhere. See picture at the end.

Problem 7.2.9: As in 7.2.5, we compute a cosine series.

$$a_0 = \frac{1}{\pi}\int_0^\pi x^2\,dx = \frac{\pi^2}{3}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx = \frac{2}{\pi n}\Big[x^2\sin nx\Big]_0^\pi - \frac{4}{\pi n}\int_0^\pi x\sin nx\,dx \\ &= \frac{4}{\pi n^2}\Big[x\cos nx\Big]_0^\pi - \frac{4}{\pi n^2}\int_0^\pi \cos nx\,dx = \frac{(-1)^n 4\pi}{n^2}, \end{aligned}$$

again resulting in the given Fourier series.

(b) Just like 7.2.5(b).

Problem 7.2.13: (a) The function is odd, so we are looking for a sine series.

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = -\frac{2}{\pi n}\Big[x\cos nx\Big]_0^\pi + \frac{2}{\pi n}\int_0^\pi \cos nx\,dx = -\frac{2}{\pi n}\pi(-1)^n = 2\frac{(-1)^{n+1}}{n}$$

once more resulting in the given Fourier series.

(b) This function is piecewise smooth, but it has a jump discontinuity at $x = \pi + 2k\pi$, $k \in \mathbb{Z}$. The series converges to the value in the middle of the jump, that is the arithmetic mean of the left and right limits, which is 0 at the jump. The convergence is non-uniform. (The partial sums of the Fourier series are continuous, so the limit would have to be continuous if the convergence was uniform.)

Problem 7.2.17: We can put $x = \pi$ in the series from problem 7.2.9. Since $\cos n\pi = (-1)^n$, we have two factors $(-1)^n$ in each term that cancel out, and we are left with

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\Big(\pi^2 - \frac{\pi^2}{3}\Big) = \frac{\pi^2}{6}.$$

Problem 7.5.11: The problem refers to Example 1 on page 493, with

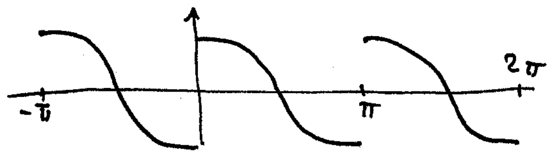
$$e^{ax} = \frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in)e^{inx} \qquad \text{for } -\pi < x < \pi.$$

In particular, for $x = 0$ the lefthand side is 1, and e^{inx} on the righthand side is also 1, so we find

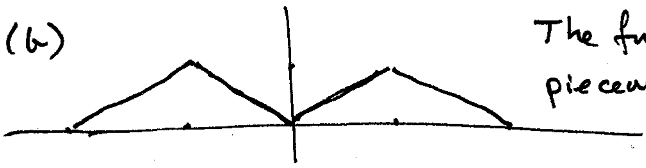
$$\frac{\pi}{\sinh \pi a} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}\,a + \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}\,in$$

where the last sum is zero because of the antisymmetry of the summed expression. Dropping that term and dividing by a we get the expansion mentioned in the problem.

7.1.12

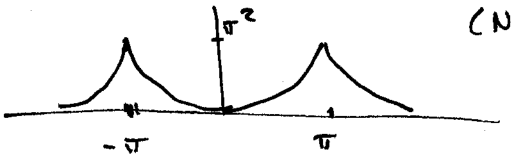


7.2.5 (b)



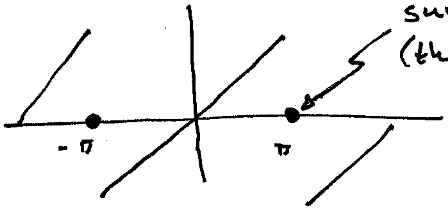
The function is continuous and piecewise smooth ...

7.2.9 (b)



(Not to scale...)

7.2.13 (b)



sum of Fourier series at $x = \pi$ (the problem did not specify f at this point)