MA2104 Fall 2006, Week 38: Solutions to exercises

Some pictures are at the end.

Problem 3.3.18: An antiderivative for the integrand is $\frac{1}{3}(z-2-i)^3 + i\ln(z-2-i) + 3(z-2-i)^{-1}$. If we choose the principal branch for the logarithm defined by $0 \le \arg z < 2\pi$, this is defined for all z except where $z-2-i$ is real and ≥ 0. In other words, everywhere except on the half line $\{2+i+t\colon t\in[0,\infty]\}$, which does not meet the integration path $C_1(0)$. Since the path is closed, the integral is therefore zero. (The principal part of the logarithm would not work here because the resulting branch cut for the $\ln(z-2-i)$ term would be the half line $\{2+i-t\colon t\in[0,\infty]\}$, which touches the integration path at one point.)

Problem 3.3.19: Using partial integration, we find an antiderivative for ze^z to be $(z-1)e^z$ (which is of course easy to verify by differentiation). This is good for all z, so we immediately find

$$\int_{[z_1,z_2,z_3]} ze^z\,dz = \Big[(z-1)e^z\Big]_{\pi}^{-1-i\pi} = (-2-i\pi)e^{-1-i\pi} - (\pi-1)e^{\pi} = (2+i\pi)e^{-1} - (\pi-1)e^{\pi}.$$

Problem 3.3.27: (a)

$$\frac{d}{dz}z^{\alpha} = \frac{d}{dz}e^{\alpha\ln z} = \frac{\alpha}{z}e^{\alpha\ln z} = \alpha e^{\alpha\ln z}e^{-\ln z} = \alpha e^{(\alpha-1)\ln z} = \alpha z^{\alpha-1}$$

provided we use the same branch of the logarithm all the way. Replacing α by $\alpha+1$ and dividing by $\alpha+1$, we find

$$\frac{d}{dz}\frac{z^{\alpha+1}}{\alpha+1} = z^{\alpha}, \qquad \text{if } \alpha \ne -1.$$

(b) Using the principal branch of the logarithm we note that the given path γ does not come near the branch cut (the negative real axis), so we can write

$$\int_{\gamma}\frac{1}{\sqrt{z}}\,dz = \int_{\gamma} z^{-1/2}\,dz = \Big[2z^{1/2}\Big]_{e^{-i\pi/2}}^{e^{i\pi/2}} = 2(e^{i\pi/4} - e^{-i\pi/4}) = 4i\sin\tfrac{\pi}{4} = 2i\sqrt{2}.$$

Problem 3.3.29: (a) Assume $|z_0| > R$. The wording is a bit funny here: What is meant is that $C_R(z_0)$ cannot intersect all four semi-axes $\{x\colon x\in[0,\infty]\}$, $\{x\colon x\in[-\infty,0]\}$, $\{iy\colon y\in[0,\infty]\}$, $\{iy\colon y\in[-\infty,0]\}$. In fact, it cannot even intersect the first two (the real semi-axes). (Quick proof: The closed ball $\bar{B}_R(z_0)$ is convex. If it contains the points $x_1\in[0,\infty]$ and $x_2\in[-\infty,0]$ then it contains 0, which is a convex combination of x_1 and x_2. But $0\notin B_R(z_0)$ since $|z_0|>R$.) So we can pick either $[-\infty,0]$ for the branch cut (i.e., use the principal branch of the logarithm) or we can use $[0,\infty]$ (i.e., use the branch defined by having the argument in $[0,2\pi)$ – this is the branch called $\log_0$ in the book, and which we might call $\ln_0$) so that we have a branch of $\ln z$ analytic on $C_R(z_0)$; and this is of course an antiderivative of $1/z$. Since the path is closed, the integral is zero.

(b) Assume $|z_0| < R$. The author of the book has really written out the proof so much that there is little to add, other than that an antiderivative of $1/z$ along γ_1 is the principal branch of the logarithm, and an antiderivative along γ_2 is the branch called $\ln_0$. So

$$\begin{aligned}\int_{\gamma}\frac{dz}{z} &= \int_{\gamma_1}\frac{dz}{z} + \int_{\gamma_2}\frac{dz}{z} = \operatorname{Ln} z_1 - \operatorname{Ln} z_2 + \ln_0 z_2 - \ln_0 z_1\\ &= \ln|z_1| + \tfrac{\pi i}{2} - (\ln|z_2| - \tfrac{\pi i}{2}) + \ln|z_2| + \tfrac{3\pi i}{2} - (\ln|z_1| - \tfrac{\pi i}{2}) = 2\pi i.\end{aligned}$$

Problem 3.3.32: The answer will of course be $2\pi i$, as we can tell by looking forward at the Cauchy integral theorem. But we can use the exact same procedure as in 3.3.29, word for word as described above.

Problem 3.3.35: (a) With notation as in the problem, when $t \to t_0$ then $\gamma(t) \to \gamma(t_0) = z_0$, so that $\epsilon(\gamma(t)) \to 0$.

(b) The stated identity is just the consequence of replacing z in $F(z) = F(z_0) + F'(z_0)(z - z_0) + \epsilon_1(z)(z - z_0)$ by $\gamma(t) = \gamma(t_0) + \gamma'(t_0)(t - t_0) + \epsilon_2(t)(t - t_0)$, and z_0 by $\gamma(t_0)$. The result is better reorganized into

$$\begin{aligned} F(\gamma(t)) = F(\gamma(t_0)) + \overbrace{F'(\gamma(t_0))\gamma'(t_0)}^{A}(t - t_0) \\ + \underbrace{\big(\epsilon_1(\gamma(t))\gamma'(t_0) + \epsilon_1(\gamma(t))\epsilon_2(t) + F'(\gamma(t_0))\epsilon_2(t)\big)}_{\epsilon(t)}(t - t_0) \end{aligned}$$

where $\epsilon(t) \to 0$ as $t \to t_0$, so that the term marked A must be the derivative of $F(\gamma(t))$ at $t = t_0$.

Problem 3.4.1: Yes: Shrink γ_0 to a point, move this point over to γ_1, and unshrink (if such a word exists) the point to γ_1.

Problem 3.4.2: No: γ_0 sorrounds two holes in the region, which γ_1 does not.

Problem 3.4.3: No: There is a hole in the way. (The answer in the back of the book is wrong.)

Problem 3.4.4: Yes: This is visually quite obvious. No holes in the way.

Problem 3.4.5: Yes: The region is convex. As always in a convex region, we can write an explicit homotopy as $H(t, s) = s\gamma_1(t) + (1 - s)\gamma_0(t)$.

Problem 3.4.6: No: The two paths surround the same hole, but in opposite directions.

Problem 3.4.10: The region in figure 36 is convex, the others are not. See picture at the end.

Problem 3.4.17: The integration path is $C_1(i)$ and the integrand is analytic in $\mathbb{C}\backslash\{-i\}$. Since the integration path is closed and $-i$ lies outside it, and the integrand is analytic, the integral is zero.

Problem 3.4.29: Perhaps the easiest thing is to do a partial fraction decomposition of the integrand? Write

$$\frac{1}{(z+1)^2(z^2+1)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

and multiply by the common denominator to get

$$\begin{aligned} 1 &= A(z+1)(z^2+1) + B(z^2+1) + C(z+i)(z+1)^2 + D(z-i)(z+1)^2 \\ &= (A + C + D)z^3 + (A + B + (2+i)C + (2-i)D)z^2 \\ &\quad + (A + (1+2i)C + (1-2i)D)z + A + B + iC - iD, \end{aligned}$$

leading to

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2+i & 2-i \\ 1 & 0 & 1+2i & 1-2i \\ 1 & 1 & i & -i \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with the solution $A = B = \frac{1}{2}$ and $C = D = -\frac{1}{4}$.

The integral of $A/(z+1)$ is $2\pi i A = \pi i$ (from integrating once around $z = -1$ in the positive direction).

The integral of $B/(z+1)^2$ is zero since the integrand has an antiderivative $-B/(z+1)$ and the integration path is closed.

The integral of $C/(z+i)$ is zero, since the point $z = -i$ lies outside the integration contour.

The integral of $D/(z-i)$ is $2\pi i D = -\frac{1}{2}\pi i$ (from integrating once around $z = i$ in the positive direction).

So the total integral is $\pi i + 0 + 0 - \frac{1}{2}\pi i = \frac{1}{2}\pi i$.

Problem 3.4.32: Again, try a partial fraction decomposition with and get

$$\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{(\alpha-\beta)(z-\alpha)} + \frac{1}{(\beta-\alpha)(z-\beta)}.$$

By Jordan's curve theorem, the simple, closed path C has just one inside and an outside. It surrounds each point inside just once in the same direction, either positive or negative. So if α is inside C, the first term on the righthand side above contributes

$$\pm\frac{2\pi i}{\alpha-\beta}$$

to the integral. Similarly, if β is inside, the second term contributes

$$\pm\frac{2\pi i}{\beta-\alpha}$$

If they are both inside, the contributions add to zero. So the possible values for the integral are zero (if none or both of α, β lie inside), or

$$\pm\frac{2\pi i}{\alpha-\beta}$$

(with the plus sign if the curve is positively oriented and only α lies inside, or if the curve is negatively oriented and only β lies inside).

Problem 3.4.33: (a) After multiplying through by the common denominator, the left hand side is 1 while the right hand side is $(A_1 + A_2 + \cdots + A_n)z^{n-1}$ plus lower order terms. So that sum must be zero.

(b) Integrating the partial fraction decomposition around C results in $\pm(A_1 + A_2 + \cdots + A_n)2\pi i = 0$.

Problem 3.4.36: (a) This is sort of obvious.

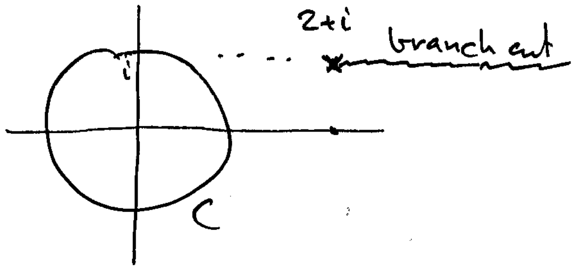
(b) Equally obvious!

(c) The new map H is well defined because the two definitions for $s = \frac{1}{2}$ agree: They are $H_1(t,1) = \gamma_1(t)$ and $H_2(t,0) = \gamma_1(t)$. Then H is continuous because H_1 and H_2 are continuous, and we are done. (We need to check that H is a homotopy with closed paths, or with fixed end points, if H_1 and H_2 are of these kinds. But this is easy too.)

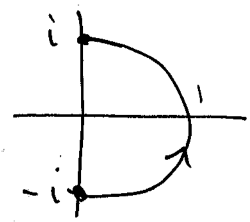
(d) Well, it's what the definition of reflexive says.

Problem 3.4.37: Something isn't quite right with this problem. I'll get back to it.

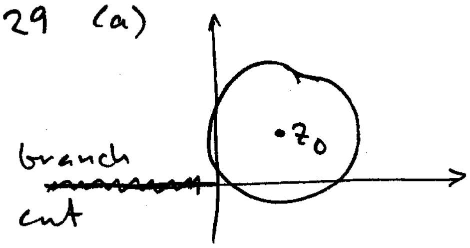
3.3.18



3.3.27



3.3.29 (a)



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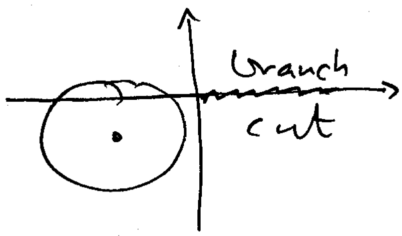
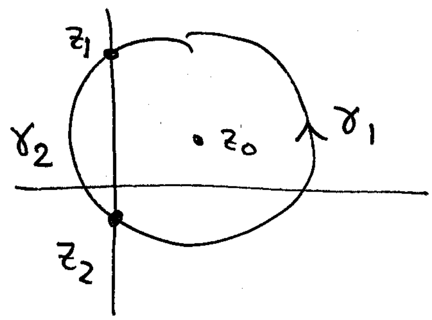
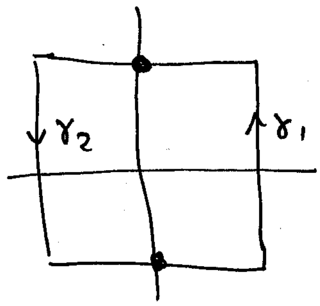


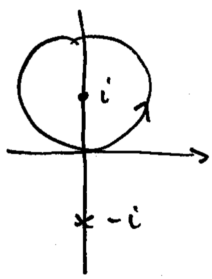
(b)



3.3.32



3.4.17



3.4.32

etc: 8 possibilities total: two orientations, α, β inside/outside.

3.4.36 (c)