MA2104 Fall 2006, Week 37: Solutions to exercises

Some pictures are at the end.

Problem 3.1.3: $z = 3i + e^{it}$ for $t \in [0, 2\pi]$ (eller et vilkårlig annet lukket intervall med lengde 2π).

Problem 3.1.6: $z = e^{-it}$ for $t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

Problem 3.1.7:

$$z = \begin{cases} it & 0 \le t \le 1, \\ 1 - t + (2-t)i & 1 \le t \le 2, \\ t - 3 & 2 \le t \le 3. \end{cases}$$

Problem 3.1.18: (Picture at end.)

Problem 3.1.23: Standard differentiation rules:

$$f'(t) = 2\frac{t+i}{t-i} \cdot \frac{t - i - (t+i)}{(t-i)^2} = -4i\frac{t+i}{(t-i)^3}.$$

Problem 3.2.3:

$$\int_{-1}^{0} \sin(ix)\,dx = \left[\frac{-\cos(ix)}{i}\right]_{x=-1}^{0} = i\bigl(\cos 0 - \cos(-i)\bigr) = i\bigl(1 - \cosh 1\bigr).$$

Problem 3.2.6: The differentiation rule $(z^\alpha)' = \alpha z^{\alpha-1}$ holds, so long as both powers are computed using the same branch of the logarithm.

In particular, we can use the standard rule for antiderivatives:

$$\int_1^2 x^i\,dx = \left[\frac{x^{1+i}}{1+i}\right]_{x=1}^{2} = \frac{2^{1+i} - 1}{1+i} = \frac{e^{(1+i)\ln 2} - 1}{1+i} = \frac{2(\cos\ln 2 + \sin\ln 2) - 1}{1+i}.$$

Since we were instructed to use the principal branch of the power, that means using the principal branch of the logarithm, which in this case means that $\ln 2$ is the real value.

Problem 3.2.11: The direct way: Parametrize $C_R(z_0)$ using $z = z_0 + Re^{it}$ for $t \in [0, 2\pi]$ so that $(\overline{z - z_0})^n = R^n e^{-int}$. Further $dz = iRe^{it}\,dt$, so we find

$$\int_{C_r(z_0)} (\overline{z - z_0})^n\,dz = \int_0^{2\pi} R^n e^{-int} \cdot iRe^{it}\,dt = iR^{n+1}\int_0^{2\pi} e^{-i(n-1)t}\,dt$$

which has the stated values.

Another way is to use the fact that $(\overline{z - z_0})(z - z_0) = |z - z_0|^2 = R^2$ for z on $C_R(z_0)$, so that

$$\int_{C_R(z_0)} (\overline{z - z_0})^n\,dz = R^{2n}\int_{C_R(z_0)} \frac{dz}{(z - z_0)^n}$$

and use the fact that we already know the value of the latter integral: It is 0 unless $n = 1$, in which case it is $2\pi i$.

Problem 3.2.16: Since $2z + i$ has the antiderivative $z^2 + iz$ and the path is closed, the integral is zero. Computed directly, it becomes

$$\int_0^{2\pi} (2e^{it} + i) \cdot ie^{it}\,dt = \int_0^{2\pi} (2ie^{2it} - e^{it})\,dt = 0.$$

Problem 3.2.17: First, with the parametrization $z = ti + (1-t) = 1 + (i-1)t$ for $t \in [0,1]$:

$$\int_{[1,i]} 2\bar{z}\,dz = 2\int_0^1 \big(1 + (-i-1)t\big)\cdot(i-1)\,dt = \big(2-(i+1)\big)(i-1) = 2i.$$

Next, with $z = t(1+i) + (1-t)i = t + i$:

$$\int_{[i,1+i]} 2\bar{z}\,dz = 2\int_0^1 (t-i)\,dt = 1 - 2i.$$

Adding the integrals together, we have

$$\int_{[1,i,1+i]} 2\bar{z}\,dz = 1.$$

Problem 3.2.21: The integrand has the antiderivative $\frac{1}{2}z^2$, and the path is closed, so the integral is zero. You were perhaps not supposed to know this yet when this exercise was due, so a direct computation is

$$\int_0^1 x\,dx + \int_0^1 (1+iy)\cdot i\,dy - \int_0^1 (x+i)\,dx - \int_0^1 iy\cdot i\,dy = \tfrac{1}{2} + (i - \tfrac{1}{2}) - (\tfrac{1}{2} + i) + \tfrac{1}{2} = 0.$$

Problem 3.2.26: I interpret the problem as having $\gamma(t) = 8e^{it} + 5e^{-4it}$ for $t \in [0, 2\pi]$. Thus

$$\begin{aligned}\int_\gamma \bar{z}\,dz &= \int_0^{2\pi} (8e^{-it} + 5e^{4it})\cdot(8ie^{it} - 20ie^{-4it})\,dt \\ &= i\int_0^{2\pi} (64 - 100 + 40e^{3it} - 160e^{-5it})\,dt \\ &= -72i\pi.\end{aligned}$$

Problem 3.2.31: $\gamma(t) = \frac{1}{5}t^5 + \frac{i}{4}t^4$ implies $\gamma'(t) = t^4 + it^3$, so $|\gamma'(t)| = \sqrt{t^8 + t^6} = t^3\sqrt{t^2+1}$. Therefore the length of the curve is (substituting $u = t^2$, which gives $du = 2t\,dt$):

$$\begin{aligned}\int_0^1 t^3\sqrt{t^2+1}\,dt &= \tfrac{1}{2}\int_0^1 u\sqrt{u+1}\,du = \tfrac{1}{2}\int_0^1 (u+1-1)\sqrt{u+1}\,du \\ &= \tfrac{1}{2}\int_0^1 \big((u+1)^{3/2} - (u+1)^{1/2}\big)\,du \\ &= \tfrac{1}{2}\big(\tfrac{2}{5}(2^{5/2}-1) - \tfrac{2}{3}(2^{3/2}-1)\big) = \tfrac{2}{15}\big(\sqrt{2}+1\big)\end{aligned}$$

Problem 3.2.40: From the definition of the integral as limits of sums: Each summand has a factor $z_j - z_{j-1} = 0$, so the whole sum is zero.

Or from the definition for piecewise smooth paths: $\gamma'(t) = 0$ along the whole parameter interval means we're integrating the constant zero.

Problem 3.3.8: $\frac{1}{2}e^{z^2} - \ln z$ is an antiderivative, valid in any region in which we can define a continuous branch of the logarithm.

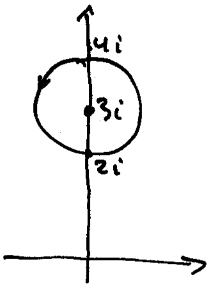
Problem 3.3.11: Even partial integration is allowed in the search for an antiderivative. This results in the candidate $\frac{1}{2}z^2\,\mathrm{Ln}\,z - \frac{1}{4}z^2$, and differentiation shows that it is indeed an antiderivative.

Problem Extra: Find all the values of i^i?

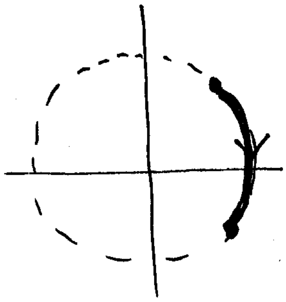
Now $\ln i = \ln|i| + i\arg i = 0 + i\frac{\pi}{2} + 2ik\pi$ with $k \in \mathbb{Z}$, which yields

$$i^i = e^{i\ln i} = e^{-\pi/2 + 2k\pi}, \qquad k \in \mathbb{Z}.$$

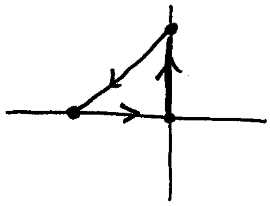
3.1.3



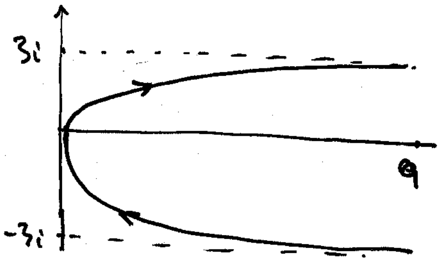
3.1.6



3.1.7

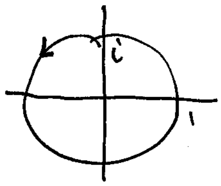


3.1.18

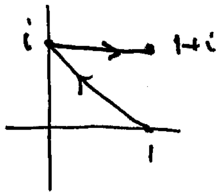


A HORRIBLE DRAWING! IT'S A PARABOLA, SYMMETRIC ABOUT THE REAL AXIS.

3.2.16



3.2.17



3.2.21