MA2104 Fall 2006, Week 34: Solutions to exercises

Some pictures are at the end.

Problem 1.2.12:

$$\Big|\frac{1+i}{(1-i)(1+3i)}\Big| = \frac{|1+i|}{|1-i|\cdot|1+3i|} = \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{10}} = \frac{1}{\sqrt{10}}$$

Problem 1.2.23: Some people are having difficulties here because the problem is *too easy*, and they think there must be more to it! Write as usual $z = x + iy$ with $x, y \in \mathbb{R}$ below. (a) $\operatorname{Re} z = a$ becomes $x = a$. (b) $\operatorname{Im} z = b$ becomes $y = b$. (c) It's just a repeat of the standard parametrization of a line in the plane. But *if you really wish to work harder*, you can write $z_2 - z_1$ on polar form as $z_2 - z_1 = re^{i\theta}$ and note that then $z = z_1 + t(z_2 - z_1)$ can be multiplied by $e^{-i\theta}$ to become $e^{-i\theta}z = e^{-i\theta}z_1 + tr$, from which you can take the imaginary part to get $\operatorname{Im}(e^{-i\theta}z) = b$, where $b = \operatorname{Im}(e^{-i\theta}z_1)$ is constant. So this says that multiplying by $e^{-i\theta}$ maps the given curve to a horizontal line. But that is just rotating by an angle $i\theta$, and a line rotated is still a line.

Problem 1.2.24: My preferred solution: The given equation is equivalent to $z_1 - z_2 = t(z_1 - z_3)$ with $t \in \mathbb{R}$, which says that $z_1 - z_2$ and $z_1 - z_3$ are parallel. And elementary geometry says that happens if and only if the points are on a line.

But if you wish to use 1.2.23(c), note that that problem concluded that z_3 is on the line through z_1 and z_2 if and only if we can write $z_3 = z_1 + t(z_2 - z_1)$ with $t \in \mathbb{R}$. But that equation is equivalent to $z_3 - z_1 = t(z_2 - z_1)$, which can be rewritten as $z_1 - z_2 = t^{-1}(z_1 - z_3)$, which is of the same form as the given equation with t replaced by t^{-1}. Since $t_3 \neq t_2$ by assumption then $t = 0$ or $t^{-1} = 0$ in these equations is not a worry.

Problem 1.2.38: We are given $|z - i| \leq \frac{1}{2}$ and asked for an upper bound to $1/|z-1|$. That is, we are asked for the constant c in an inequality $1/|z-1| \leq c$. That inequality is equivalent to $|z-1| \geq 1/c$, se we are looking for an *lower* bound on $|z-1|$.

A bit of geometric reasoning can help: The given inequality says that z is no farther than $\frac{1}{2}$ from i. The distance from i to 1 is $\sqrt{2}$, and the closest z can get to 1 is by being on that line, at a distance of $\sqrt{2} - \frac{1}{2}$.

Or, using the triangle inequality:

$$|1-i| = |1 - z + z - i| \leq |1-z| + |z-i|$$

so that $|z-1| \geq \sqrt{2} - |z-i| \geq \sqrt{2} - \frac{1}{2}$. So we get the upper bound

$$\Big|\frac{1}{1-z}\Big| \leq \frac{1}{\sqrt{2}-\frac{1}{2}} = \frac{2}{2\sqrt{2}-1}\cdot\frac{2\sqrt{2}+1}{2\sqrt{2}+1} = \frac{2}{2\sqrt{2}-1} = \frac{4\sqrt{2}+2}{7}.$$

(Pick the simplest answer for yourself.)

Problem 1.3.34: $z^3 = 1 + i$ can be written on polar form with $z = re^{i\theta}$ as $r^3e^{3i\theta} = \sqrt{2}e^{i\pi/4}$, from which $r^3 = \sqrt{2}$ (so $r = 2^{1/6}$) and $3\theta = \frac{1}{4}\pi + 2k\pi$, with $k \in \mathbb{Z}$[1] In other words $\theta = \frac{1}{12}\pi + 2k\pi/3$, but only the values $k = 0, 1, 2$ yield different z. Final solution: z must be one of

$$2^{1/6}e^{i\pi/12}, \quad 2^{1/6}e^{9i\pi/12}, \quad 2^{1/6}e^{17i\pi/12}.$$

The principal root is the first of these.

[1] $\mathbb{Z}$ is the set of integers.

Problem 1.3.36: Same idea as above, but now $-30 = 30e^{i\pi}$, and we get the solutions

$$z = 30^{1/5}e^{i\pi/5}, \quad z = 30^{1/5}e^{3i\pi/5}, \quad z = 30^{1/5}e^{5i\pi/5} = -30^{1/5},$$
$$z = 30^{1/5}e^{7i\pi/5}, \quad z = 30^{1/5}e^{9i\pi/5}.$$

Again the principal root is the first, although the middle root (at the end of the first line) seems like a more natural root when you're used to real variables.

Problem 1.3.43: Just use the standard formula for quadratic equations! Or complete the square:

$$\begin{gathered}
z^2 - 2(1+i)z + i = 0 \\
z^2 - 2(1+i)z + (1+i)^2 = -i + (1+i)^2 = 1 + i + i^2 = i \\
(z-1-i)^2 = i \\
z - 1 - i = \pm\sqrt{i} = \pm\frac{1}{2}\sqrt{2}(1+i) \\
z = 1 + i \pm \frac{1}{2}\sqrt{2}(1+i)
\end{gathered}$$

and you can write the final answers in various forms.

Problem 1.3.48: De Moivre's identity, for third powers, in extremely shortened version, is $(e^{i\theta})^3 = e^{3i\theta}$, or written out:

$$\begin{gathered}
(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \\
\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta
\end{gathered}$$

(the coefficients on the left are i^0, $3i^1$, $3i^2$, i^3), and it's now just a question of comparing imaginary parts to solve the problem (and real parts to do 1.3.47).

Problem 1.3.51: Write $z = re^{i\theta}$. Then $z^n = 1$ becomes $r^n e^{in\theta} = 1$, which implies $r = 1$ and $n\theta = 2k\pi$ with $k \in \mathbb{Z}$. Write the latter as $\theta = 2k\pi/n$ and note that adding a whole multiple of 2π to θ does not change z, so only $k = 0, 1, 2, \ldots, n-1$ produce different values. We write

$$\omega_k = e^{2ik\pi/n}, \qquad k = 0, 1, \ldots, n-1$$

(or $k = 1, 2, \ldots n$ as the book says). $\omega_0 = 1$ (ω_n if you wish) is the trivial root, and also the principal root.

(The *primitive* roots are the ω_k for which k and n have no nontrivial factors in common. Equivalently, a primitive root of $z^n = 1$ is a number ω so that every root of the equation is some integral power of ω. In particular, if n is prime then all nontrivial roots are primitive.)

Problem 1.3.53: Now we get into a bit of a terminology problem. Let me write just ω for one of the ω_k with $k \in \{1, 2, \ldots, n-1\}$.

(a) Trivial: $(\omega\omega_j)^n = \omega^n\omega_j^n = 1 \cdot 1 = 1$.

(b) Also trivial: Multiply $\omega_j \neq \omega_k$ by the nonzero ω.

(c) By (b), the roots are both $\{\omega_1, \ldots, \omega_n\}$ and $\{\omega\omega_1, \ldots, \omega\omega_n\}$. Summing, we get

$$\omega_1 + \cdots + \omega_n = \omega\omega_1 + \cdots + \omega\omega_n = \omega(\omega_1 + \cdots + \omega_n)$$

so $\omega_1 + \cdots + \omega_n = 0$ because $\omega \neq 1$.

(d) This is much prettier:

$$(1-\omega)(1 + \omega + \cdots + \omega^{n-1}) = 1 + \omega + \cdots + \omega^{n-1} - (\omega + \omega^2 + \cdots + \omega^n) = 1 - \omega^n = 0.$$

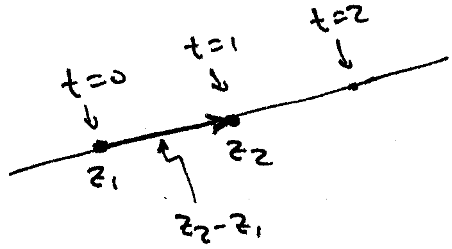
Since $(1-\omega) \neq 0$, the sum is zero. You recognize the derivation of the formula for the sum of a finite geometric series here, right?

Problem 1.4.24: Notice that multiplying by i rotates S by 90 degrees in the positive direction, while adding 2 moves the result two units to the right. See the drawings further back.

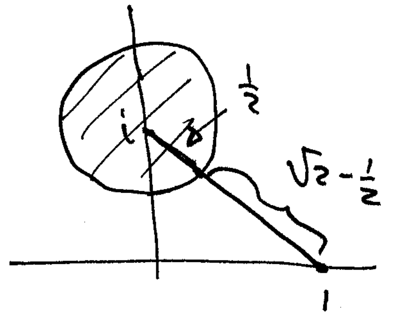
Problem 1.5: *Find the image* $f[S]$ *of the set* $S = \{x + iy\colon x \le 0 \text{ and } -\pi \le y \le 0\}$ *where* $f(z) = e^z$*.*

Now $f(x+iy) = e^{x+iy} = e^x(\cos y + i \sin y)$. $x \le 0$ translates into $0 < e^x \le 1$, and $-\pi \le y \le 0$ places the point in the lower halfplane. The image is a half disk $0 < |w| \le 1$, $\operatorname{Im} w \le 0$. (Note that the origin is not part of the image.) See drawings further back.

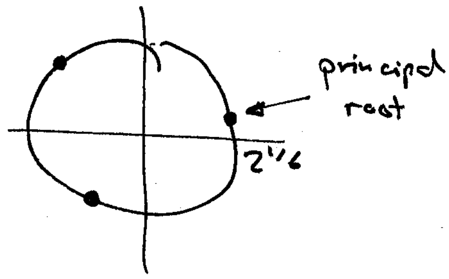
1.2.23



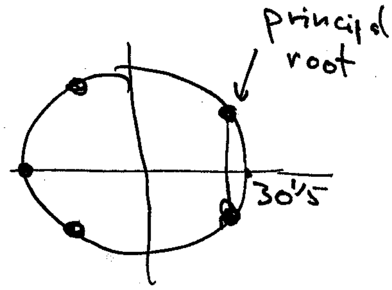
1.2.38



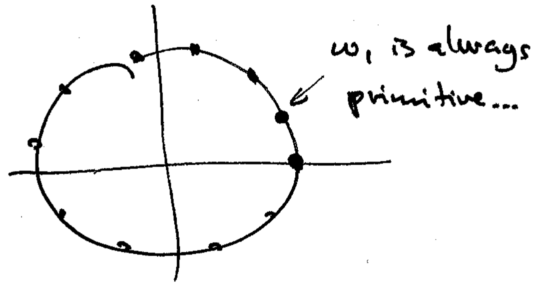
1.3.34



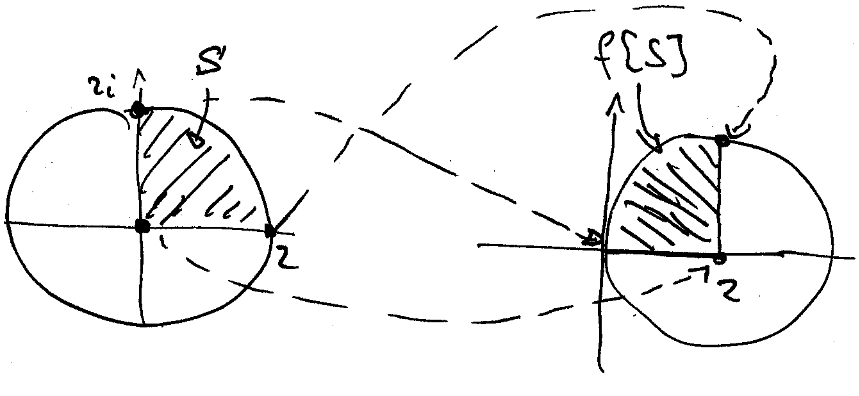
1.3.36



1.3.51



1.4.24



1.5.