**STUDFORSK PROJECT REPORT**

Project name: Algebraic cell decompositions via torus actions
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## 1. Introduction and an example

When analyzing different spaces, their cell decompositions make concrete calculations of algebraic invariants efficient. For example, cell structures can be used for topological spaces, and Morse theory help us achieve this for smooth manifolds.

In algebraic geometry, the central objects of study are varieties, i.e. the zeros of polynomials in $\mathbb{C}^n$. These varieties are the analogue of manifolds in Morse theory, and so algebraic Morse theory is the analogous study which helps us to understand the cell decomposition of varieties.

Such an algebraic Morse theory has been studied by Białynicki-Birula in [1]. The rough idea is as follows: We let the nonzero complex numbers $\mathbb{C}^*$ act on the zero set $Z(f)$ of a polynomial $f$ by multiplication in each coordinate. If the set of fixed points under this action is in some sense "nice", then $Z(f)$ has an algebraic cell structure, and we compute the dimension of the cells by studying the fixed points.

The first goal of this project is to apply algebraic Morse theory to compute the cell structure of the Grassmannian $\mathrm{Gr}(k, n)$ of $k$-planes in $n$-dimensional affine space $\mathbb{C}^n$. We achieve this goal in the following section. We then move on and look at a much more complicated case, the Hilbert scheme of $n$ points in the $d$-dimensional affine complex space $\mathbb{C}^d$. We apply the cell decomposition of the Grassmannian to compute the dimension of this Hilbert scheme. Finally we look at the tangent space of the Hilbert scheme. Our initial hope was to find a cell structure on the Hilbert scheme as well, but this ambitious goal has not been achieved in this report.

We now start our computations. As a first example, we begin with complex projective space $\mathbb{P}(n)$ on which we let $\mathbb{C}^*$ act by

$$z \cdot [x_0, x_1, \ldots, x_n] = [x_0, zx_1, z^2x_2, \ldots, z^nx_n],$$

for the point $[x_0, x_1, \ldots, x_n]$ in $\mathbb{P}^n$.

Since regular scalar multiplication does not move points anywhere, the fixed points under the action are the $n+1$ points $F_0 = [1\colon 0\colon \ldots \colon 0]$, $F_1 = [0\colon 1\colon 0\colon \ldots \colon 0]$, $\ldots$ with $F_i$ being the point with $x_i = 1$ and all other coordinates being 0. If we try to find the Byalynicki-Birula decomposition (see [1]), we need to figure out the spaces $X_i$ with

$$X_i = \{x \in \mathbb{P}^n | \lim_{z\to 0} z \cdot x = F_i\}.$$

For $i = 0$, we need $x_0 = 1$ which remains unchanged under the $\mathbb{C}^*$-action. The other coordinates are arbitrary, since the limit of $z^j x_j$ will be 0 when $z$ goes to 0. Hence

$$X_0 = \{x \in \mathbb{P}^n | x_0 = 1\} = \{[1, x_1, x_2, \ldots, x_n] \in \mathbb{P}^n | (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n\} \cong \mathbb{A}^n.$$

For $i = 1$, we need $x_0 = 0$ since it does not change with $z$. Now take a point $x = [0\colon 1\colon x_2\colon \dots : x_n]$. When we let $z$ act we get

$$z \cdot x = [0\colon z\colon z^2 x_2\colon z^3 x_3\colon \dots : z^n x_n] = [0\colon 1\colon z x_2\colon z^2 x_3\colon \dots : z^{n-1} x_n],$$

since $z \in \mathbb{C} \setminus \{0\}$. Now when $z \to 0$, the coordinates $z^{j-1} x_j$ with $j \geq 2$ go to 0 and the point $zx$ tends to $F_1$. Hence

$$X_1 = \{x \in \mathbb{P}^n | x_0 = 0, x_1 = 1\} \cong \mathbb{A}^{n-1}.$$

Continuing this way, we get $X_i \cong \mathbb{A}^{n-i}$ and the decomposition of $\mathbb{P}^n$ into affine cells.

This decomposition agrees with the one determined by the split of the tangent spaces at the fixed points into spaces with negative and positive weight. The tangent space of $\mathbb{P}^n$ at the fixed point $F_i$ is given by space of linear homomorphisms $\mathrm{Hom}(F_i, \mathbb{C}^{n+1}/F_i)$. The action of $\mathbb{C}^*$ on the line corresponding to $F_i$ is by multiplication by $z^i$ (remember we start with coordinate $x_0$). Hence the $\mathbb{C}^*$-action on $\mathrm{Hom}(F_i, \mathbb{C}^{n+1}/F_i)$ sends a homomorphism $\alpha\colon F_i \to \mathbb{C}^{n+1}/F_i$ to the homomorphism

$$\begin{aligned} z \cdot \alpha\colon t \mapsto{} & z^{-i}(\alpha_0(t), z^1 \alpha_1(t), \dots, z^{i-1}\alpha_{i-1}(t), z^{i+1}\alpha_{i+1}(t), \dots, z^n \alpha_n(t)) \\ &= (z^{-i}\alpha_0(t), z^{-i+1}\alpha_1(t), \dots, z^{-1}\alpha_{i-1}(t), z^1 \alpha_{i+1}(t), \dots, z^{n-i}\alpha_n(t)). \end{aligned}$$

Hence the subspaces of negative weights are generated by the homomorphisms $t \mapsto (0, \dots, \alpha_j(t), 0, \dots)$ with $\alpha_j(t)$ in the $j$th entry and $j < i$, and the subspaces of positive weights are generated by the homomorphisms $t \mapsto (0, \dots, \alpha_j(t), 0, \dots)$ with $j > i$. The space of all positive weights in $T_{F_i}\mathbb{P}^n$ consists of the elements in $\mathrm{Hom}(F_i, \mathbb{C}^{n+1}/F_i)$ of the form $t \mapsto (0, \dots, 0, \alpha_{i+1}(t) \dots, \alpha_n(t))$. The dimension of this space is $n - i$.

## 2. A Generalization

We have seen the cell decomposition of $\mathbb{P}^n$, and since $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$, we could hope that the cell decomposition of $\mathrm{Gr}(k, n)$ has a similar calculation. So now, let $k$ be and $n$ arbitrary. For $\mathbb{P}^n$, the fixed points were the lines $F_i$, and here the fixed points are those $k$-planes where $n - k$ coordinates are zero. Hence there are $\binom{n}{n-k}$ many fixed points. We denote the $k$-plane in $\mathbb{C}^n$ where only the coordinates $x_{i_1}, \dots, x_{i_k}$ can be nonzero by $F_{i_1,\dots,i_k}$. Since these planes are determined by the vanishing of coordinates, we can use the standard basis vectors $e_{i_1}, \dots, e_{i_k}$ to represent the $k$-plane $F_{i_1,\dots,i_k}$. For example, we represent the $k$-plane $F_{1,\dots,k}$, i.e., the plane where the last $n - k$ coordinates are zero, by the first $k$ standard basis vectors $e_1, \dots, e_k$. Now we need to work out the shape of the corresponding $X_{i_1,\dots,i_k}$.

First observe that if we have a vector $v = (v_1, \dots, v_n)$, then $z \in \mathbb{C}^*$ acts on it as $z \cdot v = (v_1, z v_2, z^2 v_3, \dots, z^{n-1} v_n)$. If $v_1 = 0$, then we can rescale $zv$, because $z \cdot v$ and the vector $(0, v_2, z v_3, \dots, z^{n-2} v_n)$ span the same line in $\mathbb{C}^n$. In particular, the vector $e_i$ and $z e_i$ generate the same line, or in other words, “represent the same basis vector”. (That’s why the $F_{i_1,\dots,i_k}$ are $\mathbb{C}^*$-fixed points.)

This shows that each $k$-plane $X$ generated by basis vectors $w_1, \dots, w_k$, where each $w_i$ agrees with $e_i$ in the first $k$ coordinates and has arbitrary coordinates in the remaining $n - i$ entries, satisfies $z \cdot X \to_{z \to 0} F_{1,\dots,k}$, since the last $n - k$ coordinates all tend to 0 when $z \to 0$. Since $F_{1,\dots,k}$ is isomorphic to $\mathbb{C}^k$, we can think of the remaining $n - k$ coordinates in the $k$ basis vectors as a way to map $k$-vectors into $\mathbb{C}^{n-k}$.

This shows that $X_{1,\dots,k}$ is isomorphic to the space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{n-k})$ of linear maps from $\mathbb{C}^k$ to $\mathbb{C}^{n-k}$ and is hence an affine space of dimension $k(n-k)$. This agrees with the general result that $X_{1,\dots,k}$ is isomorphic to the tangent space of $\mathrm{Gr}(k,n)$ at $F_{1,\dots,k} \cong \mathbb{C}^k$ which is $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^n/\mathbb{C}^k) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^{n-k})$.

An arbitrary fixed point $F_{i_1,\dots,i_k}$ is represented by the standard basis vectors $e_{i_1}, \dots, e_{i_k}$. We consider these vectors as row vectors. The planes in $X_{i_1,\dots,i_k}$ then correspond to all the matrices with $k$ rows and $n-k$ columns in row echelon form with the basis vectors $e_1, \dots, e_k$ of $\mathbb{C}^k$ (!) as the column vectors in position $i_1, \dots, i_k$. For example, elements in $X_{2,4,6} \subset \mathrm{Gr}(3,7)$ correspond matrices of the form

$$\begin{pmatrix} 0 & 1 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

In the example, the dimension of $X_{2,4,6} \subset \mathrm{Gr}(3,7)$ is 6. In general, the dimension of $X_{i_1,\dots,i_k} \subset \mathrm{Gr}(k,n)$ is given as in Schubert calculus.

Let us look at the $\mathbb{C}^*$-action on tangent spaces and the corresponding decomposition. The tangent space at the fixed point $F_{i_1,\dots,i_k}$ is the space $\mathrm{Hom}(F_{i_1,\dots,i_k}, \mathbb{C}^n/F_{i_1,\dots,i_k})$. Its elements can be represented by matrices with $n$ rows and $k$ columns with row $i_1$ of the form $(1,0,\dots,0)$, row $i_2$ of the form $(0,1,0,\dots,0)$ and so forth, and row $i_k$ of the form $(0,\dots,0,1)$. For example, $F_{2,4,6}$ in $\mathrm{Gr}(3,7)$ a matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 1 & 0 \\ a_{51} & a_{52} & a_{53} \\ 0 & 0 & 1 \\ a_{71} & a_{72} & a_{73} \end{pmatrix}.$$

In general, the action of $\mathbb{C}^*$ on the column vectors of such a matrix is just the $\mathbb{C}^*$-action on elements in $\mathbb{C}^n$. But since we mod out $F_{1,\dots,k}$ in the image, we need to twist this action. The $\mathbb{C}^*$-action on the first column is twisted by multiplying with the inverse of $z^{i_1-1}$ (the inverse of the action on the $i_1$th coordinate which we mod out with $F_{1,\dots,k}$). That means that $z \in \mathbb{C}^*$ acts on the $j$th coordinate in the first column by multiplication with $z^{j-1}/z^{i_1-1} = z^{j-i_1}$. The action on the seond column is twisted by multiplying with the inverse of $z^{i_2-1}$, i.e., $z \in \mathbb{C}^*$ acts on the $j$th coordinate in the second column by multiplication with $z^{j-1}/z^{i_2-1} = z^{j-i_2}$. In general, the action of $z \in \mathbb{C}^*$ on the $j$th coordinate in the $s$th column is given by multiplication with $z^{j-1}/z^{i_s-1} = z^{j-i_s}$.

Now the tangent space $T_{F_{i_1,\dots,i_k}}\mathrm{Gr}(k,n) = \mathrm{Hom}(F_{i_1,\dots,i_k}, \mathbb{C}^n/F_{i_1,\dots,i_k})$ is the direct sum of space $T^{j,s}$ of matrices with the rows specified by $F_{1,\dots,k}$ and otherwise just one nonzero entry $a_{js}$ in position $j, s$ $(s \neq i_1, \dots, i_k)$. For example, the subspace

$T^{5,3}$ of $T_{F_{2,4,6}}\mathrm{Gr}(3,7)$ consists of matrices of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{53} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have just learned that $z \in \mathbb{C}^*$ acts on such a matrix by multiplication with $z^{j-i_s}$. Hence the subspaces with positive weight are the spaces $T^{j,s}$ of such matrices with $j > i_s$.

Note that the sum of those spaces of positive weight corresponds to the space of matrices in row echelon form described in the example above. Overall, we see that we produced the decomposition of the Grassmannian into its Schubert cells.

## 3. Hilbert Schemes

Let $\mathbb{C}^d$ stand for the $d$-dimensional affine space, so $\mathbb{C}[x_1, \dots, x_d]$ is endowed.

A subscheme in an affine scheme $\operatorname{Spec} A$ corresponds to a quotient $M$ of $A$, i.e., a surjective homomorphism $A \to M$. Let us assume $A = \mathbb{C}[X_1, \dots, X_d]$ (then $M$ is a quotient of a free module). We are interested in the Hilbert scheme of $n$ points, i.e., we are looking at ideals $I \subset \mathbb{C}[X_1, \dots, X_d]$ such that $\dim_{\mathbb{C}} \mathbb{C}[X_1, \dots, X_d]/I = n$. (A good introduction to Hilbert Schemes is Dori Bejleri's notes on the geometry of the Hilbert scheme of points on the affine space $\mathbb{C}^2$; see also [2].) Note that if we have $n$ distinct closed points, then we are looking at the sum of the one dimensional skyskraper sheaves at each point.

Since we can multiply elements in $M$ with the $X_i$'s, $M$ is a representation of $A$ and each $X_i$ corresponds to an endomorphisms of $M$. Hence specifying an endomorphism $\mathbb{C}[X_1, \dots, X_d] \to M$ corresponds to giving $d$ endomorphisms of $\mathbb{C}^n$.

Observe that the difference between the free *associative* algebra over $\mathbb{C}$ with $d$ generators, $\mathrm{Free}_k$, and $\mathbb{C}^d$ is that the generators commute in $\mathbb{C}^d$.

Note also that $\mathrm{GL}(n)$ acts on the $X_i$'s simultaneously, i.e., for $g \in \mathrm{GL}(n)$,

$$g(X_1, \dots, X_k) = (gX_1g^{-1}, \dots, gX_kg^{-1}).$$

We start with the case $k = 1$: We have the map

$$\varphi\colon \mathrm{Hilb}(\mathrm{Free}_1, n) \to S^n\mathbb{C} = \mathbb{C}^n,\ (X, v) \mapsto\ n\text{-tuple of eigenvalues of } X.$$

We can construct a map $\psi$ in the other direction as follows: Given an $n$-tuple $(z_1, \dots, z_n)$ of complex numbers, we form the polynomial

$$P(T) := \prod_i (T - z_i) = c_0 = c_1T + \dots + c_{n-1}T^{n-1} + T^n$$

which has exactly the given numbers as zeroes. This polynomial has an associated companion matrix given by

$$\begin{pmatrix} 0 & 0 & 0 & \dots & -c_0 \\ 1 & 0 & 0 & \dots & -c_1 \\ 0 & 1 & 0 & \dots & -c_2 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix}$$

which we denote by $X$. The vector $v = (1, 0, \ldots, 0)$ is then a cyclic vector, since $Xv = (0, 1, 0, \ldots), \ldots, X^{n-1}v = (0, \ldots, 0, 1)$. Hence $\{v, Xv, \ldots, X^{n-1}v\}$ forms a basis of $\mathbb{C}^n$. The maps $\varphi$ and $\psi$ are mutually inverses to each other: If $(X, v)$ is a pair with $v$ cyclic for $X$, then the matrix of $X$ in the basis $v, Xv, \ldots, X^{n-1}v$ is of the form where the $c_i$'s are the coefficients of the characteristic polynomial of $X$. The vector $v$ is sent to the vector $(1, 0, \ldots, 0)$ under this base change. Since base change corresponds to conjugation by an element in $\mathrm{GL}(n)$, the class of $(X, v)$ in $\mathrm{Hilb}(\mathrm{Free}_1, n)$ is sent to itself under $\psi \circ \varphi$. The composition $\varphi \circ \psi$ is the identity of $\mathbb{C}^n$, since the characteristic polynomial of the above matrix is exactly $P(T)$ which means that the eigenvalues remain unchanged. In particular, we get

$$\dim_{\mathbb{C}} \mathrm{Hilb}(\mathrm{Free}_1, n) = n.$$

For arbitrary $k$, let $(X_1, \ldots, X_k, v)$ be a tuple with $v \in \mathbb{C}^n$ cyclic for the $X_i$'s. The idea is that we can express the $X_i$'s in the basis given by and the required powers under the $X_i$'s. As we have seen in the case $k = 1$, this gives us $n$ free parameters to choose determining one of the endomorphisms. The remaining $k-1$ endomorphisms give us $(k-1)n^2$ many free parameters. This way we consider $\mathrm{Hilb}(\mathrm{Free}_k, n)$ as an affine fibration over $\mathrm{Hilb}(\mathrm{Free}_1, n)$ with fiber of dimension $(k-1)n^2$. This shows that $\mathrm{Hilb}(\mathrm{Free}_k, n)$ is smooth of dimension $(k-1)n^2 + n$.

To estimate the dimension of $\mathrm{Hilb}(\mathbb{C}^d, n)$, we need to determine the number of ideals $I$ in $\mathbb{C}[X_1, \ldots, X_d]$ such that $\mathbb{C}[X_1, \ldots, X_d]/I$ has dimension $n$. Since we are interested in the asymptotic behavior of the number of ideals, we can assume that there is an $r$ such that $\mathfrak{m}^{r+1} \subset I \subset \mathfrak{m}^r$. The dimension of the vector space $\mathbb{C}[X_1, \ldots, X_d]/\mathfrak{m}^r$, generated by all monomials of degree $\leq r$ changes with $r$ asymptotically like constant times $r^d$.Hence we can assume $n = r^d$ (up to a constant). By Iarrobino, the set of such ideals is in one-to-one correspondence with the linear subspaces of dimension $n$ of the $\mathbb{C}$-vector space $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ of monomials of degree $r$ in $\mathbb{C}[X_1, \ldots, X_d]$. This space has dimension $\binom{d+r-1}{d-1}$ which changes with $r$ asymptotically as a constant times $r^{d-1}$. We need to find out what the maximal number of subspaces there are (since we interested in a lower bound). The Grassmannian $\mathrm{Gr}(k, \mathfrak{m}^r/\mathfrak{m}^{r+1})$ consists of all linear subspaces of dimension $k$ in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$. Its dimension is $k(\dim \mathfrak{m}^r/\mathfrak{m}^{r+1} - k)$. This number is maximal if $k = 1/2 \dim \mathfrak{m}^r/\mathfrak{m}^{r+1}$. Hence there are approximately $(r^{d-1})^2 = r^{2d-2}$ many linear subspaces (up to a constant, that's why we forget the factor $1/4$), and hence as many ideals by Iarrobino. Now we transform

$$n = r^d \Rightarrow r = n^{1/d}, \text{ and hence } r^{2d-2} = n^{\frac{2d-2}{d}} = n^{2-2/d}.$$

Hence as $n \to \infty$, the dimension of $\mathrm{Hilb}(\mathbb{C}^d, n)$ grows like a constant times $n^{2-2/d}$.

## 4. Tangent spaces of $\mathrm{Hilb}(\mathbb{C}^d)$

For a scheme $X$ over $\mathbb{C}$, the tangent space at a point $p$ is given by

$$T_pX = \{f \colon \mathrm{Spec}\,\mathbb{C}[\epsilon]/\epsilon^2 \to X | f(\mathrm{Spec}\,\mathbb{C}) = p\}.$$

For maps from $\mathrm{Spec}\,\mathbb{C}[\epsilon]/\epsilon^2$ into $Y$ with image poitn $p$ specify a tangent directions at $p$. By definition of $\mathrm{Hilb}(X)$, we have

$$\mathrm{Hom}(B, \mathrm{Hilb}(X)) = \{\mathcal{Z} \subset B \times X \text{ flat and proper over } B\}.$$

Now let $X$ be the affine space $\mathbb{C}^d = \mathbb{A}^d_{\mathbb{C}}$. Let $Z$ be a point of the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^d)$. Writing $B = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$, the tangent space of $\mathrm{Hilb}(\mathbb{C}^d)$ at $Z$ is

$$T_Z\mathrm{Hilb}(\mathbb{C}^d) = \left\{ \mathcal{Z} \subset B \times \mathbb{A}^d \text{ flat over } B \text{ with } \mathcal{Z} \cap \mathbb{A}^d = Z \right\}.$$

The subscheme $Z$ corresponds to an ideal $I \subset \mathbb{C}[X_1, \dots, X_d]$. We write $\mathcal{O}_Z$ for the quotient ring $\mathbb{C}[X_1, \dots, X_d]/I$. Similarly, any $\mathcal{Z}$ as above corresponds to an ideal $J \subset \mathbb{C}[X_1, \dots, X_d, \epsilon]/\epsilon^2$ such that $J \cap \mathbb{C}[X_1, \dots, X_d] = I$ and $(\mathbb{C}[X_1, \dots, X_d, \epsilon]/\epsilon^2)/J$ is flat over $\mathbb{C}[\epsilon]/\epsilon^2$. We write $\mathcal{O}_{\mathcal{Z}}$ for the quotient ring $(\mathbb{C}[X_1, \dots, X_d, \epsilon]/\epsilon^2)/J$. Consider the short exaxt sequence of $\mathbb{C}[\epsilon]/\epsilon^2$-modules

$$0 \to (\epsilon) \xrightarrow{\epsilon\cdot} \mathbb{C}[\epsilon]/\epsilon^2 \to \mathbb{C} \to 0.$$

The flatness of $\mathcal{O}_{\mathcal{Z}}$ over $\mathbb{C}[\epsilon]/\epsilon^2$ implies that tensoring this sequence with $\mathcal{O}_{\mathcal{Z}}$ yields again a short exact sequence

$$0 \to (\epsilon)\mathcal{O}_{\mathcal{Z}} \xrightarrow{\epsilon\cdot} \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C} \to 0.$$

Both the left and the right hand terms are isomorphic to $\mathcal{O}_Z$ as $\mathbb{C}[X_1, \dots, X_d]$-module. On the one hand, taking the tensor product of $\mathbb{C}$ over $\mathbb{C}[\epsilon]/\epsilon^2$ means that we mod out $\epsilon$ and since $J \cap \mathbb{C}[X_1, \dots, X_d] = I$ this means

$$\mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C} \cong \mathbb{C}[X_1, \dots, X_d]/I = \mathcal{O}_Z.$$

On the other hand, the ideal $(\epsilon)\mathcal{O}_{\mathcal{Z}}$ is isomorphic to $\mathcal{O}_Z$, for if we multiply an element in in $\mathcal{O}_{\mathcal{Z}}$ which is in the image of

$$\mathbb{C}[X_1, \dots, X_d, \epsilon] \setminus \mathbb{C}[X_1, \dots, X_d]$$

under the quotient map, becomes 0 in $\mathbb{C}[X_1, \dots, X_d, \epsilon]/\epsilon^2$ (or its quotient by $J$). Hence the above exact sequence is canonically isomorphic to

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_Z \to 0.$$

This shows that any deformation $\mathcal{Z}$ of $Z$ corresponds to an extension of $\mathcal{O}_Z$ by $\mathcal{O}_Z$ over $\mathcal{O}_X = \mathbb{C}[X_1, \dots, X_d]$. This defines an isomorphism

$$T_Z\mathrm{Hilb}(\mathbb{C}^d) = \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z).$$

Applying $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_Z)$ to the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

(where $I = \mathcal{I}_Z$) yields a long exact sequence

$$\dots \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \xrightarrow{\iota^*} \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \xrightarrow{\varphi} \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to \dots$$

Here $\iota$ denotes the inclusion $I \subset \mathbb{C}[X_1, \dots, X_d]$. The term

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = H^1(X, \mathcal{O}_Z) = 0$$

vanishes, since $Z$ is affine (or we use $H^1(X, \mathcal{O}_Z) = H^1(X, \mathcal{O}_Z(s))$ for all $s \in \mathbb{N}$ and hence must be zero by Serre's vanishing theorem). Moreover, the map $\iota^*$ is 0. For, if $\alpha \in \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z)$, i.e., $\alpha \colon \mathbb{C}[X_1, \dots, X_d] \to \mathbb{C}[X_1, \dots, X_d]/I$, then

$$\iota^*(\alpha) = \alpha \circ \iota \colon I \subset \mathbb{C}[X_1, \dots, X_d] \xrightarrow{\alpha} \mathbb{C}[X_1, \dots, X_d]/I.$$

This shows $\iota^*(\alpha) = 0$. Hence this shows that the map $\varphi$ in the above long exact sequence is an isomorphism. Summarizing, we have proved

$$T_Z\mathrm{Hilb}(\mathbb{C}^d) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z).$$

## 5. Further Calculations

Throughout, we will cite [3].

Since $T_Z\mathrm{Hilb}(X,n) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$, we know from (3.4.7) of [3] that $\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) = 0$ and hence

$$\chi(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) - \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z).$$

Now (3.4.7) also tells us $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ and by (3.4.8) this is equal to $\chi(\mathcal{O}_Z, \mathcal{O}_X)$. Hence we get

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) = \chi(\mathcal{I}_Z, \mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{O}_X).$$

The short exaxt sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

yields the equation $\chi(\mathcal{I}_Z, \mathcal{O}_Z) = \chi(\mathcal{O}_X, \mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z)$. Putting this together, we get

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) = \chi(\mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{O}_X).$$

Now applying $\chi(\mathcal{O}_Z, -)$ to the sequence right above, we get

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{I}_Z) = \chi(\mathcal{O}_Z, \mathcal{O}_X)$$

and hence

$$\chi(\mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{O}_X) = \chi(\mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{I}_Z).$$

Applying $\chi(\mathcal{O}_X, -)$ and $\chi(\mathcal{I}_Z, -)$ to the same sequence, we get, respectively,

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_X, \mathcal{O}_Z) = \chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(\mathcal{O}_X, \mathcal{I}_Z) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X, \mathcal{I}_Z), \text{ and}$$
$$\chi(\mathcal{O}_Z, \mathcal{I}_Z) = \chi(\mathcal{O}_X, \mathcal{I}_Z) - \chi(\mathcal{I}_Z, \mathcal{I}_Z).$$

Overall, this gives us the second equation of the proposition:

$$\begin{aligned}\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) &= \chi(\mathcal{O}_Z) - \chi(\mathcal{O}_Z, \mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{O}_X)\\ &= \chi(\mathcal{O}_Z) + \chi(\mathcal{O}_Z, \mathcal{I}_Z)\\ &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X, \mathcal{I}_Z) + \chi(\mathcal{O}_X, \mathcal{I}_Z) - \chi(\mathcal{I}_Z, \mathcal{I}_Z)\\ &= \chi(\mathcal{O}_X) - \chi(\mathcal{I}_Z, \mathcal{I}_Z).\end{aligned}$$

With this, and the fact that $T_Z\mathrm{Hilb}(\mathbb{C}^d) = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$, one could hope that this could lead to a decomposition of $\mathrm{Hilb}(\mathbb{C}^d)$ that agrees with the one determined by the split of the tangent spaces at the fixed points into spaces with negative and positive weight. But further investigation is needed.

## References


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