On topological invariants for algebraic cobordism

27th Nordic Congress of Mathematicians, Celebrating the 100th anniversary of Institut Mittag-Leffler

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joint work with Michael J. Hopkins
Point of departure: Poincaré, Lefschetz, Hodge...
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For a differential form $\alpha$ write $\alpha \in A^{p,q}(X)$ if

$$\alpha = \sum_j f_j \, dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$
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If $\Gamma = Z$ happens to be an algebraic subvariety of $X$, say of complex dimension $n$, then

$\int_Z \iota^* \alpha = 0$ unless $\alpha$ lies in $\mathbb{A}^{n,n}(X)$.
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The Hodge Conjecture: The map

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switch to \( \mathbb{Q} \)-coefficients
A short digression: the Jacobian of a curve
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Jacobi Inversion Theorem: The (Abel–Jacobi) map

\[\mu: \text{Div}^0(C) \to \mathbb{C}^g / \Lambda =: J(C)\]

is surjective.

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Lefschetz’s proof for (1,1)-classes:
For simplicity, let $X \subset \mathbb{P}^N$ be a surface.
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A “normal function” $\nu$ is a holomorphic section of $\pi$. 
Lefschetz’s proof continued:

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Let $D$ be an algebraic curve on $X$. It intersects $C_t$ in points $p_1(t),..., p_d(t)$. 

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Choose a point $p_0$ on all $C_t$. Then $\sum_i p_i(t) - dp_0$

is a divisor of degree 0 and defines a point

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Hence $D$ defines a normal function

$$\nu_D: t \mapsto \nu_D(t) \in J.$$
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- Every normal function \( \nu \) defines a class \( \eta(\nu) \in H^2(X;\mathbb{Z}) \) of Hodge type \( (1,1) \) such that \( \eta(\nu_D) = \text{cl}_H(D) \).
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- Every class in $H^2(X;\mathbb{Z})$ of Hodge type $(1,1)$ arises as $\eta(\nu)$ for some normal function $\nu$. 
Griffiths: Higher dimensions

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Then \(\left( \omega \mapsto \int_{\Gamma} \omega \right) \in \mathcal{F}^{n-p+1} H^{2n-2p+1}(X;\mathbb{C})^\vee\).
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$X$, a smooth projective complex variety with $\dim X = n$.

$Z \subset X$ a subvariety of codimension $p$ which is the boundary of a differentiable chain $\Gamma$.

Then $\left( \omega \longmapsto \int_{\Gamma} \omega \right) \in F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})^\vee$.

But the value depends on the choice of $\Gamma$. 

The intermediate Jacobian of Griffiths and the Abel-Jacobi map:
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We obtain a well-defined map

\[ Z \mapsto \int_{\Gamma} \quad \text{for some } \Gamma \text{ with } Z = \partial \Gamma \]

\[ \mu: Z^p(X)_h \to F^{n-p+1}H^{2n-2p+1}(X;C)^{\vee}/H_{2n-2p+1}(X;Z) \]
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\( J^{2p-1}(X) \) is a complex torus and is called Griffiths’ intermediate Jacobian.
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Have an induced a map:

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\text{Griff}^p(X) := \frac{Z^p(X)_h}{Z^p(X)_{\text{alg}}} \rightarrow \frac{J^{2p-1}(X)}{J^{2p-1}(X)_{\text{alg}}}
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The Jacobian and Griffiths’ theorem:

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Griffith’s theorem: Let \( X \subset P^4 \) be a general quintic hypersurface. There are lines \( L \) and \( L' \) on \( X \) such that \( \mu(L-L') \) is a non torsion element in \( J^3(X) \).
An interesting diagram:
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$Z^p(X)$ \[\subset\] $X$

$\cl_H$

$Hdg^{2p}(X)$
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Let \( X \) be a smooth projective complex variety.

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\begin{align*}
Z^p(X) & \quad Z \subset X \\
\text{cl}_H & \quad [Z_{sm}] \\
& \quad \text{Hdg}^{2p}(X)
\end{align*}
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An interesting diagram:

Let $X$ be a smooth projective complex variety.

$Z^p(X)_h = \text{Kernel of } cl_H \subset Z^p(X) \quad Z \subset X$

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\[ \text{Abel–Jacobi map } \mu \]

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\[ J^{2p-1}(X) \quad \text{cl}_H \quad [Z_{sm}] \quad Hdg^{2p}(X) \]
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$\mathcal{J}^{2p-1}(X) \to H_{D}^{2p}(X; \mathbb{Z}(p)) \to \text{Hdg}^{2p}(X)$
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Deligne cohomology combines topological with Hodge theoretic information
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Deligne cohomology combines topological with Hodge theoretic information
An interesting diagram:

Let $X$ be a smooth projective complex variety.

$Z^p(X)_h = \ker \text{cl}_H \subset Z^p(X)$ \quad $Z \subset X$

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$0 \rightarrow J^{2p-1}(X) \rightarrow H_D^{2p}(X;\mathbb{Z}(p)) \rightarrow \text{Hdg}^{2p}(X) \rightarrow 0$

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Deligne cohomology combines topological with Hodge theoretic information
Another interesting map for smooth complex varieties:
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$$\Phi : \Omega^*(X) \rightarrow \mathcal{MU}^{2*}(X)$$
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\( \Omega^p(X) \) is generated by projective maps \( f: Y \rightarrow X \) of codimension \( p \) with \( Y \) smooth variety modulo Levine's and Pandharipande's "double point relation":

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$\Omega^p(X)$ is generated by projective maps $f: Y \rightarrow X$ of codimension $p$ with $Y$ smooth variety modulo Levine’s and Pandharipande’s “double point relation”:

$\pi^{-1}(0) \sim \pi^{-1}(\infty)$ for projective morphisms $\pi: Y' \rightarrow X \times \mathbb{P}^1$ such that $\pi^{-1}(0)$ is smooth and $\pi^{-1}(\infty) = A \cup_D B$ where $A$ and $B$ are smooth and meet transversally in $D$. 
What can we say about the map $\Phi$?

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$[Y \to X]$

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- The image:

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\Phi & \\
\Omega^*(X) & \quad \xrightarrow{\Phi} \quad MU^{2*}(X)
\end{align*}
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  $[Y \to X] \xlongleftarrow{\Phi} [Y(C) \to X(C)]$

- The image:

  $\Omega^*(X) \xrightarrow{\Phi} MU^{2*}(X)$

$Z^*(X)/_{\text{rat.eq}} = CH^*(X)$
What can we say about the map $\Phi$?

- The image:
  $$\Omega^*(X) \xrightarrow{\Phi} \text{MU}^{2*}(X)$$

  $$\mathbb{Z}^*(X)_{\text{rat.eq}} = \text{CH}^*(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X;\mathbb{Z})$$
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There is a “Hodge-theoretic” restriction for $\text{Im} \Phi$. 
What can we say about the map $\Phi$?

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$$
\text{Im}\Phi \quad \text{Hdg}^{2*}(X) \subseteq \text{H}^{2*}(X;\mathbb{Z})
$$

- The kernel:

$$
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$$
What can we say about the map $\Phi$?

- The image: $\Omega^\bullet(X) \xrightarrow{\Phi} \text{MU}^{2\bullet}(X)$

- The kernel:

  Griffiths’ theorem suggests that $\Phi$ is not injective.

There is a “Hodge-theoretic” restriction for $\text{Im}\Phi$. 
What can we say about the map $\Phi$?

• The image:

$$\begin{align*}
\Omega^*(X) & \xrightarrow{\Phi} \text{MU}^{2*}(X) \\
\Omega^*(X) & \xrightarrow{\Phi} \text{CH}^*(X) \\
\Omega^*(X) & \xrightarrow{\Phi} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X;\mathbb{Z})
\end{align*}$$

$Z^*(X)/\text{rat.eq} = \text{CH}^*(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X;\mathbb{Z})$

There is a “Hodge-theoretic” restriction for $\text{Im}\Phi$.

• The kernel:

Griffiths’ theorem suggests that $\Phi$ is not injective.

**Question:** Is there is an “Abel–Jacobi-invariant” which is able to detect elements in $\text{Ker}\Phi$?
The image:

$$\Omega^*(X) \xrightarrow{\Phi} \text{MU}^2(X)$$
The image:

\[ \Omega^*(X) \xrightarrow{\Phi} Hdg_{MU^{2*}}(X) \cap MU^{2*}(X) \]
The image: not surjective, but ...

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\[ \Omega^*(X) \xrightarrow{\Phi} Hdg_{MU^2*}(X) \cap MU^2*(X) \]

\[ \Omega^*(X) \otimes_{L*} Z \xrightarrow{} MU^2*(X) \otimes_{L*} Z \]
The image: not surjective, but ...
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\[ \Omega^* (X) \xrightarrow{\Phi} \text{Hdg}_{MU^2*}(X) \]

\[ \Omega^* (X) \otimes_{L^*} \mathbb{Z} \xrightarrow{\Phi} \text{MU}^{2*}(X) \otimes_{L^*} \mathbb{Z} \]

\[ \text{CH}^* (X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subset \text{H}^{2*}(X; \mathbb{Z}) \]
The image: not surjective, but ...

\[ \Omega^*(X) \xrightarrow{\Phi} \text{Hdg}_{MU^2}(X) \]

\[ \Omega^*(X) \otimes_{L^* Z} \rightarrow \text{MU}^2(X) \]

\[ \text{CH}^*(X) \xrightarrow{\text{cl}_H} \text{Hdg}^2(X) \subseteq H^2(X;Z) \]
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\[ \Omega^*(X) \xrightarrow{\Phi} \text{Hdg}_{\text{MU}^2}(X) \cap \text{MU}^{2*}(X) \]

\[ \Omega^*(X) \otimes_{L^*} \mathbb{Z} \xrightarrow{\text{Totaro}} \text{MU}^{2*}(X) \otimes_{L^*} \mathbb{Z} \]

\[ \text{CH}^*(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X; \mathbb{Z}) \]
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\Omega^*(X) & \xrightarrow{\Phi} \text{MU}^{2*}(X) \\
\Omega^*(X) \otimes_{L^*} \mathbb{Z} & \xrightarrow{} \text{MU}^{2*}(X) \otimes_{L^*} \mathbb{Z} \\
\text{CH}^*(X) & \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X;\mathbb{Z}) \\
\end{align*}
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The image: not surjective, but ...

\[ \Omega^*(X) \xrightarrow{\Phi} MU^{2*}(X) \]

\[ \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\Phi} MU^{2*}(X) \otimes_{\mathbb{L}} \mathbb{Z} \]

Levine-Morel \( \cong \) Totaro

\[ CH^*(X) \xrightarrow{\text{cl}_H} \text{Hdg}^{2*}(X) \subseteq H^{2*}(X;\mathbb{Z}) \]

\[ Hdg_{MU}^{2*}(X) \cap \]
The image: not surjective, but...

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Levine-Morel \( \approx \) Totaro \( \neq \) in general

Atiyah-Hirzebruch: \( cl_H \) is not surjective.
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This argument does not work for $\Phi$.
Kollar's examples: (see also Soulé-Voisin et. al.)
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Let \( X \subset \mathbb{P}^4 \) a very general hypersurface of degree \( d=p^3 \) for a prime \( p \geq 5 \).

\[
H^2(X; \mathbb{Z})=\mathbb{Z} \cdot h, \quad H^4(X; \mathbb{Z})=\mathbb{Z} \cdot \alpha, \quad \int_X \alpha \cdot h=1
\]
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both torsion-free and all classes are Hodge classes
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This implies: \( \alpha \) is not algebraic (since we needed a curve of degree 1).
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But $d\alpha$ is algebraic (for $\int_X d\alpha \cdot h = d = \int_X h^2 \cdot h \Rightarrow d\alpha = h^2$).
Consequences for $\Phi: \Omega^*(X) \rightarrow MU^{2*}(X)$:
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Let $X \subset \mathbb{P}^4$ be a very general hypersurface as above.
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Then $MU^4(X) \to H^4(X;\mathbb{Z})$ is surjective, and thus Kollar’s argument implies that $\Phi$ is not surjective (on Hodge classes).
Consequences for $\Phi: \Omega^*(X) \to \text{MU}^{2*}(X)$:

Let $X \subset \mathbb{P}^4$ be a very general hypersurface as above.

Then $\text{MU}^4(X) \to \text{H}^4(X;\mathbb{Z})$ is surjective, and thus Kollar’s argument implies that $\Phi$ is not surjective (on Hodge classes).

These examples are “not topological”: there is a dense subset of hypersurfaces $Y \subset \mathbb{P}^4$ such that the generator in $\text{H}^4(Y;\mathbb{Z})$ is algebraic.
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.
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$\Omega^p(X)$
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\Omega^p(X) \quad [Y \to X]
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\[ \Omega^p(X) \xrightarrow{\Phi} \text{Hdg}_{MU}^{2p}(X) \]
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

$$\Omega^p(X) \rightarrow [Y \rightarrow X]$$

$\Phi$

$$[Y(C) \rightarrow X(C)]$$

$\text{Hdg}_{MU}^{2p}(X)$
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

$\Omega^p(X) \to^{[Y \to X]} [Y(C) \to X(C)]$

$\Phi$

$\newcommand\Hdg{\text{Hdg}}$

$\Hdg^{2p}_{MU}(X)$
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

$\Omega^p(X) \to [Y \to X]$

$\Phi$

$[Y(C) \to X(C)]$

$\text{MU}_D^{2p}(p)(X) \to \text{Hdg}_{\text{MU}}^{2p}(X)$

combines topol. cobordism with Hodge theoretic information
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

$\Omega^p(X) \rightarrow \Phi \rightarrow [Y(C) \rightarrow X(C)]$

$\mathcal{M}_{U_D}^{2p}(p)(X) \rightarrow \text{Hdg}_{\mathcal{M}_{U}}^{2p}(X) \rightarrow 0$

combines topol. cobordism with Hodge theoretic information.
A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

\[ \Omega^p(X) \xrightarrow{[Y \to X]} \]

\[ \Phi\downarrow \]

\[ [Y(C) \to X(C)] \]

\[ 0 \to J_{\text{MU}}^{2p-1}(X) \to \text{MU}_D^{2p}(p)(X) \to \text{Hdg}_{\text{MU}}^{2p}(X) \to 0 \]

combines topol. cobordism with Hodge theoretic information
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complex torus $\approx MU^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z}$
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A new diagram: (joint work with Mike Hopkins)

Let $X$ be any smooth projective complex variety.

$\Omega^p(X)_{\text{top}} := \text{Kernel of } \Phi \subset \Omega^p(X) \quad [Y \to X]$

$0 \to J_{\text{MU}}^{2p-1}(X) \to \text{MU}_D^{2p}(p)(X) \to \text{Hdg}_{\text{MU}}^{2p}(X) \to 0$

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"Abel-Jacobi map" $\mu_{MU}$

$$0 \to J_{MU}^{2p-1}(X) \to MU_D^{2p}(p)(X) \to \text{Hdg}_{MU}^{2p}(X) \to 0$$

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The Abel-Jacobi map:
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$\text{Sing} \cdot \text{MU}_n(X)$

$K(F_p A^*(X; V^*), n) \rightarrow Z^n(X \times \Delta^\bullet; V^*)$
The Abel-Jacobi map: Given $n, p$

$$\text{Simpl. map. space}$$

$$\text{Sing} \cdot \text{MU}_n(X)$$

$$K(F^p A^*(X; V^*), n) \rightarrow \mathbb{Z}^n(X \times \Delta^*; V^*)$$
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Sing•MU_n(X)
The Abel–Jacobi map: Given \( n, p \)

\[ V^* := \text{MU}^* \otimes C \]

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cocycles
The Abel-Jacobi map: Given $n, p$

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$$\text{Sing} \cdot \text{MU}_n(X)$$

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**EM-space**

**cocycles**
The Abel-Jacobi map: Given $n$, $p$

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space

$$\text{MU}_n(p)(X) \longrightarrow \text{Sing} \bullet \text{MU}_n(X)$$

htpy. cart.

$$K(F^pA^*(X;V^*),n) \longrightarrow \mathbb{Z}^n(X \times \Delta^k; V^*)$$

EM-space
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EM-space cocycles
The Abel–Jacobi map: Given \( n, p \)

Elements in \( \text{MU}_D^n(p)(X) \) consist of \((f, h, \omega)\): space

\[
\pi_0 \text{MU}_n(p)(X) \to \text{Sing} \cdot \text{MU}_n(X)
\]

htpy. cart.

\[
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\]

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The Abel–Jacobi map: Given \( n, p \)

Elements in \( \text{MU}_D^n(p)(X) \) consist of \((f, h, \omega)\): 

- \( f : X \to \text{MU}_n \)

\[ V^* := \text{MU}^* \otimes C \]

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\[ \text{space} \]

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The Abel–Jacobi map: Given \( n, p \)

Elements in \( MU_D^n(p)(X) \) consist of \( (f, h, \omega) \):

- \( f : X \to MU_n \)
- \( \omega \in F^pA^n(X;V^*) \)

\( V^* := MU^* \otimes C \)

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- \( f : X \to \text{MU}_n \)
- \( \omega \in F^pA^n(X;V^*) \)
- \( h \in C^{n-1}(X;V^*) \)
  
such that “\( \partial h = f - \omega \)"
The Abel-Jacobi map: Given \( n, p \)

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If \( n = 2p \), \( [f] = 0 \) and \( [\omega] = 0 \), then \( (f, h, \omega) \) defines an element in \( \text{MU}^{2p-1}(X) \otimes R \), uniquely modulo \( \text{MU}^{2p-1}(X) \).
The Abel-Jacobi map: Given $n$, $p$

Elements in $\text{MU}_D^n(p)(X)$ consist of $(f, h, \omega)$: space

- $f : X \to \text{MU}_n$
- $\omega \in F^pA^n(X;V^*)$
- $h \in C^{n-1}(X;V^*)$

such that $\partial h = f - \omega$

If $n=2p$, $[f]=0$ and $[\omega]=0$, then $(f, h, \omega)$ defines an element in $\text{MU}^{2p-1}(X) \otimes \mathbb{R}$, uniquely modulo $\text{MU}^{2p-1}(X)$.

This gives the Abel-Jacobi map

$$\Omega^p(X)_{\text{top}} \to \text{MU}^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z}$$
The Abel–Jacobi map: Given $n, p$

Elements in $\text{MU}_D^n(p)(X)$ consist of $(f, h, \omega)$:

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- $h \in \text{C}^{n-1}(X; V^*)$

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If $n=2p$, $[f]=0$ and $[\omega]=0$, then $(f, h, \omega)$ defines an element in $\text{MU}^{2p-1}(X) \otimes \mathbb{R}$, uniquely modulo $\text{MU}^{2p-1}(X)$.

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\[ \Omega^p(X)_{\text{top}} \to \text{MU}^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z} \]
Examples:

The new Abel-Jacobi map is able to detect interesting algebraic cobordism classes:

\[ \Omega^p(X) \]

\[ \text{CH}^p(X) \quad J_{MU}^{2p-1}(X) \quad \text{MU}^{2p}(X) \]
Examples:

The new Abel–Jacobi map is able to detect interesting algebraic cobordism classes:

\[ \exists \alpha \in \Omega^p(X) \]

\[ \begin{array}{ccc}
\text{CH}^p(X) & \xrightarrow{\mu_{MU}} & \text{MU}^{2p}(X) \\
& \downarrow \Phi & \\
\text{J}_{MU}^{2p-1}(X) & &
\end{array} \]
Examples:

The new Abel–Jacobi map is able to detect interesting algebraic cobordism classes:

\[ \exists \alpha \in \Omega^p(X) \]

\[ \begin{array}{c}
0 \\
\text{CH}^p(X) \\
\text{J}^{2p-1}_{\text{MU}}(X) \\
\text{MU}^{2p}(X)
\end{array} \]

\[ \begin{array}{c}
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\mu_{\text{MU}} \\
\end{array} \]
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\[ \begin{array}{ccc}
0 & \xrightarrow{\mu_{\text{MU}}} & 0 \\
\text{CH}^p(X) & \xrightarrow{\Phi} & \text{MU}^{2p}(X)
\end{array} \]
Examples:

The new Abel–Jacobi map is able to detect interesting algebraic cobordism classes:

\[ \exists \alpha \in \Omega^p(X) \]

\[
\begin{array}{ccc}
0 & \overset{\phi}{\longrightarrow} & \mathcal{J}_{\text{MU}}^{2p-1}(X) \\
\mathcal{C}H^p(X) & \overset{\mu_{\text{MU}}}{\longrightarrow} & \text{MU}^{2p}(X)
\end{array}
\]
Thank you!