LECTURE NOTES ON CHARACTERISTIC CLASSES, K-THEORY AND THE ADAMS CONJECTURE

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Contents

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Preface

These are my incomplete lecture notes from the class MATH231br *Advanced Algebraic Topology* that I taught in spring 2014 at Harvard University. The goal of the class was to give an introduction to the powerful theory of characteristic classes on real and complex vector bundles, to introduce complex K-theory, and to provide a first outlook on some of the fascinating interactions of these theories with homotopy theory. The course consisted of three classes per week, in total 36 classes which explains the number of sections. The notes are often quite informal as they were written to be used as actual notes during classes. For example, the content of one section may recalled in another even though the reader may not feel the necessity for such a recollection.

Moreover, with the exception of some minor corrections, the notes have not been changed or updated after the classes in 2014. Unfortunately, this also means that the notes are incomplete and some topics that were discussed in class are missing in this file, since we do not have a written documentation of those classes.

However, as additional material, I added a collection of slides on the Adams conjecture in an appendix. The slides are compiled from an invited lecture series in Heidelberg on étale homotopy theory in March 2014. As an application of the étale homotopy type of Artin–Mazur and Friedlander, I briefly discussed the proofs of Friedlander, Quillen, and Sullivan of the Adams conjecture. Again, the slides have not been updated or corrected for typos since 2014.

Despite all these shortcomings I hope that these notes are useful nevertheless.

I plan to improve and update these notes in the future and would be happy to receive comments and suggestions for improvements at any time. Please send them to gereon.quick@ntnu.no.

Gereon Quick

1. Vector bundles

We start with the basic theory of vector bundles. For the moment there is nothing special about the complex case, we could also consider real vector bundles. Later, when we define K-theory, it will, however, matter if we work with complex or real bundles. Our references for the next lectures are the book of Milnor and Stasheff and Hatcher's online notes.

We introduce the first main character of the story.

Definition 1.1. Let B be a topological space.

1) A *family of real vector spaces* ξ over B consists of the following data:

- a topological space $E = E(\xi)$ called the *total space*
- a continuous $\pi\colon E \to B$ called the *projection map*, and
- for each $b \in B$ the structure of a vector space over the real numbers $\mathbb{R}$ in the set $E_b := \pi^{-1}(b)$.

2) The family ξ is called a *real vector bundle* over B if these data are subject to the following condition:

- *Local triviality*: For each point $b \in B$ there should exist a neighborhood $U \subset B$, an integer $n \geq 0$, and a homeomorphism
$$h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$$
such that, for each $b \in U$, the correspondence $x \mapsto h(b,x)$ defines an isomorphism between the vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$.

3) A *family of complex vector spaces* ζ over B consists of the data:

- a topological space $E = E(\zeta)$ called the *total space*
- a continuous $\pi\colon E \to B$ called the *projection map*, and
- for each $b \in B$ the structure of a vector space over the complex numbers $\mathbb{C}$ in the set $\pi^{-1}(b)$.

4) The family ζ is called a *complex vector bundle* over B if these data are subject to the following condition:

- *Local triviality*: For each point $b \in B$ there should exist a neighborhood $U \subset B$, an integer $n \geq 0$, and a homeomorphism
$$h\colon U \times \mathbb{C}^n \to \pi^{-1}(U)$$
such that, for each $b \in U$, the correspondence $z \mapsto h(b,z)$ defines an isomorphism between the vector space $\mathbb{C}^n$ and the vector space $\pi^{-1}(b)$.

For vector bundles, we will use some further terminology:

- A pair (U,h) as in Definition 1.1 will be called a *local trivialization* about b.
- If it is possible to choose U equal to the entire space B of a vector bundle, then the vector bundle will be called a *trivial bundle*.
- We often refer to a vector bundle $\pi : E \to B$ by just mentioning the total space E.
- The vector space $\pi^{-1}(b)$ is called the *fiber* over b. It will also be denoted by E_b.
- The fiber $E_b = \pi^{-1}(b)$ is never vacuous, but it may consist of a single point. The dimension n of E_b is allowed to vary, but it is always a *locally constant* function. Though in most cases of interest the dimension is constant. In this case one speaks of an *n-dimensional bundle* and call n the *rank* of the bundle.
- A 1-dimensional bundle is also called a *line bundle*.

Now that we have the basic notions at hand, we will focus for a while on real vector bundles and we will often refer to a real vector bundle just as a vector bundle. Later, when we introduce K-theory we will look at complex bundles again.

So let us have a look at some examples of (real) vector bundles.

Example 1.2. There is an obvious example of a vector bundle over any topological space B: The *product* or *trivial* bundle $E = B \times \mathbb{R}^n$ with π the projection onto the first factor.

Example 1.3. Let $I = [0,1]$ be the unit interval, and let E be the quotient space of $I \times \mathbb{R}$ under the identification $(0,t) \sim (1,-t)$. Then the projection $I \times \mathbb{R} \to I$ induces a map
$$\pi\colon E \to S^1$$

which is a line bundle. Since E is homeomorphic to a Möbius band, i.e., a cylinder cut open, twisted once and glued back together, with its boundary circle deleted, we call this bundle the *Möbius bundle*.

Example 1.4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. The *tangent bundle τ to S^n* is the vector bundle $\pi : E \to S^n$ where

$$E = \{(x,v) \in S^n \times \mathbb{R}^n | \ x \perp v\}.$$

We think of v as a tangent vector to S^n by translating it so that its tail is at the head of x on S^n. The map $\pi \colon E \to S^n$ sends (x,v) to x.

The vector space structure on $\pi^{-1}(x)$ is defined by

$$t_1(x,v_1) + t_2(x,v_2) = (x, t_1 v_1 + t_2 v_2).$$

In order to show that this is a vector bundle we have to construct local trivializations. So let $x \in S^n$ be any point and let $U_x \subset S^n$ be the open hemisphere which contains x and is bounded by the hyperplane through the origin orthogonal to x.

Define

$$h_x \colon \pi^{-1}(U_x) \to U_x \times \pi^{-1}(x) \cong U_x \times \mathbb{R}^n$$

by

$$h_x(y,v) = (y, p_x(v))$$

where p_x is the orthogonal projection onto the hyperplane $\pi^{-1}(x)$. PICTURE!

Then h_x is a local trivialization, since p_x restricts to an isomorphism of $\pi^{-1}(y)$ onto $p^{-1}(x)$ for each $y \in U_x$.

Example 1.5. The *normal bundle ν to S^n in* $\mathbb{R}^{n+1}$ is the line bundle $\pi \colon E \to S^n$ with E consisting of pairs

$(x,v) \in S^n \times \mathbb{R}^{n+1}$ such that v is perpendicular to the tangent plane to S^n at x,

or in other words,

$$v = tx \text{ for some } t \in \mathbb{R}.$$

DRAW A PICTURE FOR S^2!

The map $\pi \colon E \to S^n$ is just given by $\pi(x,v) = x$ and the vector space structure on $\pi^{-1}(x)$ is again defined by

$$t_1(x,v_1) + t_2(x,v_2) = (x, t_1 v_1 + t_2 v_2).$$

As in the previous example, local trivializations $h_x \colon \pi^{-1}(U_x) \to U_x \times \mathbb{R}$ can be obtained by orthogonal projection of the fibers $\pi^{-1}(y)$ onto $\pi^{-1}(x)$ for $y \in U_x$ and U_x as in the previous example.

2. Vector bundles and sections

We have seen the definition and first examples of vector bundles. Today we will first continue our list of examples. Let us get started.

Example 2.1. Recall that the real projective n-space $\mathbb{R}\mathrm{P}^n$ is the space of lines in $\mathbb{R}^{n+1}$ through the origin. Since each such line intersects the unit sphere S^n in a pair of antipodal points, we can also regard $\mathbb{R}\mathrm{P}^n$ as the quotient space of S^n in which antipodal pairs of points are identified, i.e., $\mathbb{R}\mathrm{P}^n = S^n/x \sim (-x)$. The topology of $\mathbb{R}\mathrm{P}^n$ is then the topology as a quotient of S^n. Let $\{\pm x\}$ denote the equivalence class of x in $S^n/\sim$

The *canonical line bundle* γ_n^1 *over* $\mathbb{R}P^n$ is the line bundle $\pi\colon E \to \mathbb{R}\mathrm{P}^n$ with total space

$$E(\gamma_n^1) = \{(\{\pm x\}, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} | v = tx \text{ for some } t \in \mathbb{R}\} \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}.$$

In other words, E is consisting of all pairs (ℓ, v) such that the vector v lies on the line ℓ.

The map $\pi\colon E \to \mathbb{R}\mathrm{P}^n$ is just the projection sending $(\{\pm x\}, v)$ to $\{\pm x\}$.

Now we need to find local trivializations for γ_n^1. Let $U \subset S^n$ be any open set which is small enough so as to contain no pair of antipodal points, and let U_1 denote the image of U in $\mathbb{R}\mathrm{P}^n$. Then a homeomorphism

$$h\colon U_1 \times \mathbb{R} \to \pi^{-1}(U_1)$$

is defined by the requirement that

$$h(\{\pm x\}, t) = (\{\pm x\}, tx)$$

for each $(x,t) \in U \times \mathbb{R}$. The pair (U_1,h) is a local trivialization of γ_n^1.

After seeing some examples of vector bundles we would like to be able to say when two bundles are isomorphic.

Definition 2.2. 1) Let ξ and η be two vector bundles over some base space B. Then we say that ξ *is isomorphic to* η, written $\xi \cong \eta$, if there exists a homeomorphism

$$f : E(\xi) \to E(\eta)$$

between the total spaces which maps each vector space $E_b(\xi)$ isomorphically onto the corresponding vector space $E_b(\eta)$.

2) We say that a bundle is trivial if it is isomorphic to the product bundle $B \times \mathbb{R}^n$ for some $n \geq 0$.

Example 2.3. 1) The tangent bundle τ_1 to S^1 is isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. The isomorphism is given by the map

$$\tau_1 \to S^1 \times \mathbb{R}, (e^{i\theta}, ie^{i\theta}) \mapsto (e^{i\theta}, t) \text{ for } e^{i\theta} \in S^1 \text{ and } t \in \mathbb{R}.$$

Recall that the total space of τ^1 is given by the space

$$E(\tau_1) = \{(x,v) \in S^1 \times \mathbb{R}^1 | \ x \perp v\} = \{(e^{i\theta}, ie^{i\theta}) | \ t \in \mathbb{R}, \theta \in [0,2\pi]\}.$$

Note: The triviality of τ_1 is special to the case $n = 1$. Though the situation is simpler for the normal bundle.

2) The normal bundle ν of S^n in $\mathbb{R}^{n+1}$ is isomorphic to the product line bundle $S^n \times \mathbb{R}$. The isomorphism is given by the map

$$(x,tx) \mapsto (x,t).$$

Hence ν is trivial.

But, of course, not all bundles are trivial.

Proposition 2.4. *The canonical line bundle γ_n^1 over $\mathbb{R}\mathrm{P}^n$ is not trivial for $n \geq 1$.*

We prove this claim by studying the sections of γ_n^1.

Definition 2.5. A *section* of a vector bundle $\pi \colon E \to B$ is a continuous map

$$s \colon B \to E$$

which takes each $b \in B$ into the corresponding fiber $\pi^{-1}(b)$. In other words, s is a continuous map such that $\pi \circ s = \mathrm{id}_B$.

A section is called *nowhere zero* if $s(b)$ is a non-zero vector of $\pi^{-1}(b)$ for each b.

Example 2.6.

- Every vector bundle has a *zero section* whose value is the zero vector in each fiber.
- A trivial bundle possesses a nowhere zero section.

From the last point we see that in order to proof Proposition 2.4 it suffices to show that γ_n^1 does not have nowhere zero section:

Let

$$s \colon \mathbb{R}\mathrm{P}^n \to E(\gamma_n^1)$$

be any section, and consider the composition

$$S^n \to \mathbb{R}\mathrm{P}^n \xrightarrow{s} E(\gamma_n^1)$$

which carries each $x \in S^n$ to some pair

$$(\{\pm x\}, t(x)x) \in E(\gamma_n^1).$$

Since this map is the composite of continuous maps it is itself continuous and hence the map $x \mapsto t(x)$ is a continuous map $S^n \to \mathbb{R}$, i.e. it is a continuous real valued function. Moreover, it satisfies

$$t(-x) = -t(x).$$

Since S^n is connected it follows from the intermediate value theorem that $t(x_0) = 0$ for some x_0. Hence

$$s(\{\pm x_0\}) = (\{\pm x_0\}, 0)$$

and s cannot be nowhere zero. Thus γ_n^1 is not trivial. □

Example 2.7. Let us have a closer look at the space $E(\gamma_n^1)$ for the special case $n = 1$. In this case, each point $e = (\{\pm x\}, v)$ of $E(\gamma_n^1)$ can be written as

$$e = (\{\pm(\cos\theta, \sin\theta)\}, t(\cos\theta, \sin\theta)) \text{ with } 0 \leq \theta \leq \pi, t \in \mathbb{R}.$$

This representation is unique except that for the point

$$(\{\pm(\cos 0, \sin 0)\}, t(\cos 0, \sin 0)) = (\{\pm(\cos\pi, \sin\pi)\}, -t(\cos\pi, \sin\pi)) \text{ for each } t \in \mathbb{R}.$$

In other words, $E(\gamma_n^1)$ can be obtained from the strip $[0, \pi] \times \mathbb{R}$ in the (θ, t)-plane by identifying the left hand boundary $\{0\} \times \mathbb{R}$ with the right hand boundary $\{\pi\} \times \mathbb{R}$ under the correspondence

$$(0,t) \mapsto (\pi, -t).$$

Thus $E(\gamma_n^1)$ is an open Möbius band over $\mathbb{RP}^1$. Since $\mathbb{RP}^1$ is just S^1 we see that in this case γ_1^1 is just the Möbius bundle over S^1 we defined in the previous lecture. And we see once again that γ_1^1 is non-trivial.

Another strategy to distinguish non isomorphic bundles is to look at the complement of the zero section. For any vector bundle isomorphism must the zero section to the zero section. Hence it induces a homeomorphism on the complements of the zero sections.

Example 2.8. This gives us another way to see that the Möbius bundle is nontrvival. The complement of the zero section of the Möbius bundle is connected but the complement of the zero section of the product bundle $S^1 \times \mathbb{R}$ is not connected.

3. Families of sections

We have seen in the proof that the canonical line bundle over the projective space is nontrivial that it can be very helpful to study the sections of a bundle. Today we want to push this idea a little further.

Definition 3.1. Let $\{s_1, \dots, s_n\}$ be a collection of sections of a vector bundle $\pi\colon E \to B$. The sections $s_1, \dots, s_n$ are called *nowhere linearly dependent* if, for each $b \in B$ the vectors $s_1(b), \dots, s_n(b)$ are linearly independent.

The existence of nowhere dependent sections is rather special.

Theorem 3.2. *An n-dimensional vector bundle ξ is trivial if and only if ξ admits n sections $s_1, \dots, s_n$ which are nowhere linearly dependent.*

The proof will depend on the following basic result.

Lemma 3.3. *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $E_b(\xi)$ isomorphically onto the corresponding vector space $E_b(\eta)$. Then f is necessarily a homeomorphism and ξ is isomorphic to η.*

Proof. The hypothesis on what f does with the fibers implies that f is bijective. Hence it remains to show that f^{-1} is continuous. This is a local question so let $b_0 \in B$ be any point and choose local trivializations (U,g) for ξ and (V,h) for η with $b_0 \in U \cap V$. Then we want to show that the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{h^{-1} \circ f \circ g} (U \cap V) \times \mathbb{R}^n$$

is a homeomorphism. Setting

$$h^{-1}(f(g(b,x))) = (b,y)$$

it is evident that $y = (y_1, \dots, y_n)$ can be expressed in the form

$$y_i = \sum_j f_{ij}(b) x_j$$

where $(f_{ij}(b))$ denotes an invertible $n \times n$-matrix of real numbers. Furthermore, since h^{-1}, f and g are continuous maps, the entries $f_{ij}(b)$ depend continuously on b.

Let $(F_{ji}(b))$ denote the inverse matrix. Then we have

$$g^{-1} \circ f^{-1} \circ h(b,y) = (b,x)$$

where

$$x_j = \sum_i F_{ji}(b) y_i.$$

Since the inverse of a mtrix A is given by $1/\det(A)$ times the adjoint matrix, the numbers $F_{ji}(b)$ depend continuously on the entries $f_{ij}(b)$. Hence they depend continuously on b. Thus $g^{-1} \circ f^{-1} \circ h$ is continuous. This completes the proof of the lemma □

Proof of Theorem 3.2. Let $s_1, \ldots, s_n$ be sections of ξ which are nowhere linearly dependent. Define

$$f\colon B \times \mathbb{R}^n \to E$$

by

$$f(b,x) = x_1 s_1(b) + \ldots + x_n s_n(b).$$

Evidently, f is continuous and maps each fiber of the trivial bundle ϵ_B^n isomorphically onto the corresponding fiber of ξ. The previous lemma implies that f is an isomomorphism of bundles and ξ is trivial.

Conversely, suppose that ξ is trivial, with trivialization (B,h). Defining

$$s_i(b) = h(b, (0, \ldots, 0, 1, 0, \ldots, 0)) \in E_b(\xi)$$

(with the 1 in the i-th place), it is evident that $s_1, \ldots, s_n$ are nowhere linearly dependent sections. This completes the proof of Theorem 3.2. □

Example 3.4. The tangent bundle of the circle $S^1 \subset \mathbb{R}^2$ admits one nowhere zero section

$$s(x_1, x_2) = ((x_1, x_2), (-x_2, x_1)).$$

We can rewrite this in terms of complex numbers. If we set $z = x_1 + ix_2$ then the section s is given by

$$z \mapsto iz.$$

Example 3.5. The tangent bundle to the 3-sphere $S^3 \subset \mathbb{R}^4$ admits three nowhere linearly dependent sections $s_i(x) = (x,\bar{s}_i(x))$ where

$$\begin{aligned} \bar{s}_1(x) &= (-x_2, x_1, -x_4, x_3) \\ \bar{s}_2(x) &= (-x_3, x_4, x_1, -x_2) \\ \bar{s}_3(x) &= (-x_4, -x_3, x_2, x_1). \end{aligned}$$

It is easy to check that the three vectors $\bar{s}_1(x)$, $\bar{s}_2(x)$, and $\bar{s}_3(x)$ are orthogonal to each other and to $x = (x_1, x_2, x_3, x_4)$. Hence s_1, s_2, and s_3 are nowhere linearly dependent sections of the tangent bundle of S^3 in $\mathbb{R}^4$.

The above formulas come in fact from the quaternion multiplication in $\mathbb{R}^4$. For let $\mathbb{H}$ be the quaternions, i.e., the division algebra whose elements are expressions of the form $z = x_1 + ix_2 + jx_3 + kx_4$ with $x_1, \ldots, x_4 \in \mathbb{R}$ subject to the multiplication rules

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i, \text{ and } ik = -j.$$

If we identify $\mathbb{H}$ with $\mathbb{R}^4$ via the coordinates (x_1, x_2, x_3, x_4) then we can describe the three sections s_1, s_2, and s_3 of the tangent bundle of S^3 in $\mathbb{H}$ by the formulas

$$\bar{s}_1(z) = iz$$
$$\bar{s}_2(z) = jz$$
$$\bar{s}_3(z) = kz.$$

Remark 3.6. If the tangent bundle of a manifold is trivial then one says that the manifold is parallelizable. Hence the last two examples show that S^1 and S^3 are parallelizable.

4. Constructing new bundles out of old

We already have a bunch of examples of bundles at hand. But we'd like to be able to construct new bundles out of known ones. We will see some basic constructions for new bundles today.

4.1. **Restricting a bundle to a subset of the base space.** Let ξ be a vector bundle with projection $\pi\colon E \to B$ and let U be a subset of B. Setting $E|U = \pi^{-1}(U)$, and letting

$$\pi|U\colon E|U = \pi^{-1}(U) \to U$$

be the restriction of π to $E|U$, one obtains a new vector bundle which will be denoted by $\xi|U$, and called the *restriction* of ξ to U.

Each fiber $E_b(\xi|U)$ is just equal to the corresponding fiber $E_b(\xi)$, and is given the same vector space structure.

4.2. **Induced or pullback bundles.** Let ξ be a vector bundle over B and let B_1 be an arbitrary topological space. Given a continuous map $f\colon B_1 \to B$ one can construct the *induced bundle* or *pullback bundle* $f^*\xi$ over B_1 as follows. The total space E_1 of $f^*\xi$ is the subset $E_1 \subset B_1 \times E$ consisting of all pairs (b,e) such that $f(b) = \pi(e)$, or in a formula

$$E_1 = \{(b,e) \in B_1 \times E|\ f(b) = \pi(e)\}.$$

The projection map $\pi_1\colon E_1 \to B_1$ is defined by $\pi_1(b,e) = b$. Thus one has a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{f} & B \end{array}$$

where $\hat{f}(b,e) = e$. The vector space structure in $\pi^{-1}(b)$ is defined by

$$t_1(b,e_1) + t_2(b,e_2) = (b, t_1e_1 + t_2e_2).$$

Thus $\hat{f}$ carries the vector space $E_b(f^*(\xi)$ isomorphically onto the vector space $E_{f(b)}(\xi)$.

It remains to specify the local trivializations of $f^*\xi$. If (U, h) is a local trivialization for ξ, we set $U_1 = f^{-1}(U)$ and define

$$h_1\colon U_1 \times \mathbb{R}^n \to \pi_1^{-1}(U_1) \text{ by } h_1(b,x) = (b, h(f(b), x)).$$

Then (U_1, h_1) is a local trivialization of $f^*\xi$.

Example 4.1. If ξ is trivial, then $f^*\xi$ is trivial. For if $E = B \times \mathbb{R}^n$ then the total space E_1 of $f^*(\xi)$ consists of the triples (b_1, b, x) in $B_1 \times B \times \mathbb{R}^n$ with $b = f(b_1)$. Hence b does not induce any restriction and E_1 is just the product $B_1 \times \mathbb{R}^n$.

Remark 4.2. If $f\colon B_1 \to B$ is an inclusion map, then there is an isomorphism

$$E|B_1 \cong f^*(E)$$

given by sending $e \in E$ to the point $(\pi(e),e)$.

We still have not yet said what a map between bundles over different base spaces should be. The above construction inspires the following definition.

Definition 4.3. Let ξ and η be two vector bundles. A *bundle map* from η to ξ is a continuous map

$$g\colon E(\eta) \to E(\xi)$$

which carries each vector space $E_b(\eta)$ isomorphically onto one of the vector spaces $E_{b'}(\xi)$ for some $b' \in B(\xi)$.

Remark 4.4. Setting $\bar{g}(b) = b'$, we obtain a map

$$\bar{g}\colon B(\eta) \to B(\xi).$$

This map is continuous. For $\bar{g}$ is completely determined by g, since the projection map π_η of η is surjective:

$$\begin{array}{ccc} E(\eta) & \xrightarrow{g} & E(\xi) \\ {\scriptstyle \pi_\eta}\downarrow & & \downarrow{\scriptstyle \pi_\xi} \\ B(\eta) & \xrightarrow{\bar{g}} & B(\xi). \end{array}$$

Now since the question is local, we can choose a local trivialization (U,h) of ξ. Then it suffices to prove the assertion for a map of trivial bundles and a diagram

$$\begin{array}{ccc} V \times \mathbb{R}^n & \xrightarrow{g} & U \times \mathbb{R}^n \\ {\scriptstyle \pi_\eta}\downarrow & & \downarrow{\scriptstyle \pi_\xi} \\ V & \xrightarrow{\bar{g}} & U. \end{array}$$

But now it is clear that $\bar{g}$ is continuous since g is continuous and $\bar{g}(b)$ is just the first coordinate of $g(b,x)$.

Lemma 4.5. *If $g\colon E(\eta) \to E(\xi)$ is a bundle map, and if $\bar{g}\colon B(\eta) \to B(\xi)$ is the corresponding map of base spaces, then η is isomorphic to the induced bundle $\bar{g}^*\xi$.*

Proof. Define

$$h\colon E(\eta) \to E(\bar{g}^*\xi) \text{ by } h(e) = (\pi(e), g(e))$$

where π denotes the projection map of η. Since h is continuous and maps each fiber $E_b(\eta)$ isomorphically onto the corresponding fiber $E_b(\bar{g}^*\xi)$, it follows from the lemma of the previous lecture that h is an isomorphism. □

The previous lemma shows the following uniqueness statement.

Proposition 4.6. *Given a map $f\colon B_1 \to B$ and a vector bundle ξ over B, then $f^*\xi$ is up to isomorphism the unique vector bundle ξ' over B_1 which is equipped with a map to ξ which takes the fiber of ξ' over b isomorphically onto the fiber of ξ over $f(b)$ for each $b \in B_1$.*

Moreover, the pullback construction is natural in the following sense: If we have another continuous map $g\colon B_2 \to B_1$, then there is a natural isomorphism

$$g^*f^*(\xi) \cong (f \circ g)^*(\xi)$$

given by sending each point of the form

$$(b, e) \text{ to the point } (b, g(b), e), \text{ where } b \in B_2, e \in E.$$

Conclusion 4.7. For a space B let $\mathrm{Vect}^n(B)$ denote the set of isomorphism classes of n-dimensional vector bundles over B. Then a continuous map

$$f\colon B_1 \to B$$

induces a map

$$f^*\colon \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(B_1) \text{ sending } \xi \text{ to } f^*\xi.$$

4.3. **Cartesian products.** Given two vector bundles ξ_1, ξ_2 with projection maps $\pi_i\colon E_i \to B_i$, $i = 1, 2$, the *Cartesian product* $\xi_1 \times \xi_2$ is defined to be the bundle with projection map

$$\pi_1 \times \pi_2\colon E_1 \times E_2 \to B_1 \times B_2$$

where each fiber

$$(\pi_1 \times \pi_2)^{-1}(b_1, b_2) = E_{b_1}(\xi_1) \times E_{b_2}(\xi_2)$$

is given the obvious vector space structure.

4.4. **Whitney sums.** Now let ξ_1, ξ_2 be two vector bundles over the same space B. Let

$$d\colon B \to B \times B$$

denote the diagonal embedding. The bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and will be denoted $\xi_1 \oplus \xi_2$. Each fiber $E_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum of the fibers $E_b(\xi_1) \oplus E_b(\xi_2)$.

Definition 4.8. Consider two vector bundles ξ and η over the same base space B with $E(\xi) \subset E(\eta)$. Then ξ is a *sub-bundle* of η, written $\xi \subset \eta$, if each fiber $E_b(\xi)$ is a sub-vector space of the corresponding fiber $E_b(\eta)$.

Lemma 4.9. *Let ξ_1 and ξ_2 be sub-bundles of η such that each vector space $E_b(\eta)$ is equal to the direct sum of the sub-spaces $E_b(\xi_1)$ and $E_b(\xi_2)$. Then η is isomorphic to the Whitney sum $\xi_1 \oplus \xi_2$.*

Proof. Define a map

$$f\colon E(\xi_1 \oplus \xi_2) \to E(\xi) \text{ by } f(b, e_1, e_2) = e_1 + e_2.$$

The lemma of the previous lecture shows that f is an isomorphism of bundles since it maps the fibers isomorphically onto each other. □

4.5. **Euclidian vector bundles.** Let V be a finite dimensional real vector space. Recall that a real valued function $q\colon V \to \mathbb{R}$ is called *quadratic* if q satisfies $q(av) = a^2 q(v)$ for every $v \in V$ and $a \in \mathbb{R}$ and the map $b\colon V \times V \to \mathbb{R}$ defined by

$$b(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$$

is a symmetric bilinear pairing. We also write $v \cdot w$ for $b(v,w)$. We have in particular: $v \cdot v = q(v)$. The quadratic function q is called *positive definite* if $q(v) > 0$ for every $v \neq 0$.

Definition 4.10. A *Euclidean vector space* is a real vector space V together with a positive definite quadratic function

$$q\colon V \to \mathbb{R}.$$

The real number $v \cdot w$ is called *inner product* of the vectors v and w. The number $q(v) = v \cdot v$ is also denoted by $|v|^2$.

Definition 4.11. A *Euclidean vector bundle* is a real vector bundle ξ together with a continuous map

$$q\colon E(\xi) \to \mathbb{R}$$

such that the restriction of q to each fiber of ξ is positive definite and quadratic. The map q is called a *Euclidian metric* on ξ.

In the case of the tangent bundle τ_M of a smooth manifold, a Euclidian metric $q\colon DM \to \mathbb{R}$ is called a *Riemannian metric*, and M together with q is called a *Riemannian manifold*.

Example 4.12. a) The trivial bundle ϵ^n_B on a space B can be given the Euclidean metric

$$q(b,x) = x_1^2 + \ldots + x_n^2.$$

b) Since the tangent bundle of $\mathbb{R}^n$ is trivial it follows that the smooth manifold $\mathbb{R}^n$ possesses a standard Riemannian metric. Moreover, any smooth manifold $M \subset \mathbb{R}^n$, the composition

$$DM \subset D\mathbb{R}^n \xrightarrow{q} \mathbb{R}$$

makes M into a Riemannian manifold.

Lemma 4.13. *Let ξ be a trivial bundle of dimension n over a space B and let q be any Euclidean metric on ξ. Then there exist n sections $s_1, \ldots, s_n$ of ξ which are normal and orthogonal in the sense that*

$$s_i(b) \cdot s_j(b) = \delta_{ij}$$

for each $b \in B$ where δ_{ij} is the Kronecker symbol.

Proof. The lemma of the previous lecture shows that ξ admits n nowhere dependent sections. Pointwise application of the Gram-Schmidt orthonormalization process yields orthonormal sections. □

4.6. **Orthogonal complements.** Given a sub-bundle $\xi \subset \eta$, is there a complementary sub-bundle so that η splits as a Whitney sum? If η is a Euclidean bundle, we can always find such a complement. We can construct it as follows.

Let $E_b(\xi^\perp)$ denote the subspace of $E_b(\eta)$ consisting of all vectors v such that $v \cdot w = 0$ for all $E_b(\xi)$. Let $E(\xi^\perp)$ denote the union of all $E_b(\xi^\perp)$.

Theorem 4.14. *The space $E(\xi^\perp)$ is the total space of a sub-bundle $\xi^\perp \subset \eta$, and η is isomorphic to the Whitney sum $\xi \oplus \xi^\perp$. The bundle $\xi^\perp$ is called the* orthogonal complement *of ξ in η.*

Proof. It is clear that each fiber $E_b(\eta)$ is the direct sum of the subspaces $E_b(\xi)$ and $E_b(\xi^\perp)$. Thus it remains to show the local triviality of $\xi^\perp$. The lemma of the previous lecture then implies that the map $(v,w) \mapsto v + w$ is an isomorphism of vector bundles.
Given any point $b_0 \in B$, let U be a neighborhood of b_0 which is sufficiently small that both $\xi|U$ and $\eta|U$ are trivial. Since $\xi|U$ is trivial, we can choose orthonormal sections $s_1, \ldots, s_m$ of $\xi|U$. We may enlarge this set of sections to a set of n independent local sections of $\eta|U$ by first choosing $s'_{m+1}, \ldots, s'_n$ first in the fiber $E_{b_0}(\eta)$. By the continuity of the determinant function, there is a neighborhood $V \subset U$ of b_0 such that $s_1(b), \ldots, s_m(b), s'_{m+1}(b), \ldots, s'_n(b)$ are linearly independent for all $b \in V$ and such that the $s_i(b)$ vary continuously with b in V. Applying the Gram-Schmidt orthonormalization process to $s_1, \ldots, s_m, s'_{m+1}, \ldots, s'_n$ in each fiber to obtain new sections $s_1, \ldots, s_n$. The formulae for this process show that the s_i vary continuously with $b \in V$. We can now define a trivialization

$$h\colon V \times \mathbb{R}^{n-m} \to E(\xi^\perp)$$

by the formula

$$h(b,x) = x_1 s_{m+1}(b) + \ldots + x_{n-m} s_n(b).$$

□

4.7. **Stably trivial bundles.** The direct sum of two trivial bundles is of course again trivial. But the direct sum of two nontrivial bundles can also be trivial. If one bundle is trivial, this phenomenon has been given a name.

Definition 4.15. A vector bundle ξ over B is called *stably trivial* if the direct sum $\xi \oplus \epsilon^n$ is a trivial bundle for some n.

Example 4.16. The direct sum of the tangent bundle τ and the normal bundle ν to S^{n-1} in $\mathbb{R}^n$ is the trivial bundle $S^{n-1} \times \mathbb{R}^n$. For the elements of the direct sum $\tau \oplus \nu$ are triples $(x,v,tx) \in S^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ with $x \perp v$, and the map

$$(x, v, tx) \mapsto (x, v + tx)$$

gives an isomorphism of $\tau \oplus \nu$ with $S^{n-1} \times \mathbb{R}^n$. Since the normal bundle ν is trivial, this shows that τ is stably trivial.

But there are also examples where both bundle are nontrivial whereas their Whitney sum is trivial.

Example 4.17. Let γ_n^1 be the canonical line bundle on $\mathbb{RP}^n$. Then the map $(\ell, v, w) \mapsto (\ell, v + w)$ for $v \in \ell$ and $w \perp \ell$ defines an isomorphism $\gamma_n^1 \oplus (\gamma_n^1)^\perp \cong \mathbb{RP}^n \times \mathbb{R}^{n+1}$.

Example 4.18. Specializing the previous example to the case $n = 1$, we see that

$$\gamma_1^1 \oplus (\gamma_1^1)^\perp \cong \mathbb{RP}^1 \times \mathbb{R}^2 \cong S^1 \times \mathbb{R}^2.$$

The map that rotates a vector by 90 degrees defines an isomorphism between $(\gamma_1^1)^\perp$ and γ_1^1. Since γ_1^1 is isomorphic to the Möbius bundle over S^1, this shows that the direct sum of the Möbius bundle with itself is the trivial bundle.

5. Euclidean bundles, orthogonal complements and orientation

Recall that we defined the Whitney sum of two bundles:

Let ξ_1, ξ_2 be two vector bundles over the same space B. Let

$$d\colon B \to B \times B$$

denote the diagonal embedding. The bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and will be denoted $\xi_1 \oplus \xi_2$. Each fiber $E_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum of the fibers $E_b(\xi_1) \oplus E_b(\xi_2)$.

Definition 5.1. Consider two vector bundles ξ and η over the same base space B with $E(\xi) \subset E(\eta)$. Then ξ is a *sub-bundle* of η, written $\xi \subset \eta$, if each fiber $E_b(\xi)$ is a sub-vector space of the corresponding fiber $E_b(\eta)$.

Lemma 5.2. *Let ξ_1 and ξ_2 be sub-bundles of η such that each vector space $E_b(\eta)$ is equal to the direct sum of the sub-spaces $E_b(\xi_1)$ and $E_b(\xi_2)$. Then η is isomorphic to the Whitney sum $\xi_1 \oplus \xi_2$.*

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5.1. Euclidean vector bundles.

Let V be a finite dimensional real vector space. Recall that a real valued function $q\colon V \to \mathbb{R}$ is called *quadratic* if q satisfies $q(av) = a^2 q(v)$ for every $v \in V$ and $a \in \mathbb{R}$ and the map $b\colon V \times V \to \mathbb{R}$ defined by

$$b(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$$

is a symmetric bilinear pairing. We also write $v \cdot w$ for $b(v,w)$. We have in particular: $v \cdot v = q(v)$. The quadratic function q is called *positive definite* if $q(v) > 0$ for every $v \neq 0$.

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The real number $v \cdot w$ is called *inner product* of the vectors v and w. The number $q(v) = v \cdot v$ is also denoted by $|v|^2$.

Definition 5.4. A *Euclidean vector bundle* is a real vector bundle ξ together with a continuous map

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such that the restriction of q to each fiber of ξ is positive definite and quadratic. The map q is called a *Euclidian metric* on ξ.

In the case of the tangent bundle τ_M of a smooth manifold, a Euclidian metric $q\colon DM \to \mathbb{R}$ is called a *Riemannian metric*, and M together with q is called a *Riemannian manifold*.

Example 5.5. a) The trivial bundle ϵ_B^n on a space B can be given the Euclidean metric

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Lemma 5.6. *Let ξ be a trivial bundle of dimension n over a space B and let q be any Euclidean metric on ξ. Then there exist n sections $s_1, \ldots, s_n$ of ξ which are normal and orthogonal in the sense that*

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for each $b \in B$ where δ_{ij} is the Kronecker symbol.

Proof. The lemma of the previous lecture shows that ξ admits n nowhere dependent sections. Pointwise application of the Gram-Schmidt orthonormalization process yields orthonormal sections. □

5.2. **Orthogonal complements.** Given a sub-bundle $\xi \subset \eta$, is there a complementary sub-bundle so that η splits as a Whitney sum? If η is a Euclidean bundle, we can always find such a complement. We can construct it as follows.

Let $E_b(\xi^\perp)$ denote the subspace of $E_b(\eta)$ consisting of all vectors v such that $v \cdot w = 0$ for all $E_b(\xi)$. Let $E(\xi^\perp)$ denote the union of all $E_b(\xi^\perp)$.

Theorem 5.7. *The space $E(\xi^\perp)$ is the total space of a sub-bundle $\xi^\perp \subset \eta$, and η is isomorphic to the Whitney sum $\xi \oplus \xi^\perp$. The bundle $\xi^\perp$ is called the* orthogonal complement *of ξ in η.*

Proof. It is clear that each fiber $E_b(\eta)$ is the direct sum of the subspaces $E_b(\xi)$ and $E_b(\xi^\perp)$. Thus it remains to show the local triviality of $\xi^\perp$. The lemma of the previous lecture then implies that the map $(v,w) \mapsto v + w$ is an isomorphism of vector bundles.

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$$h\colon V \times \mathbb{R}^{n-m} \to E(\xi^{\perp})$$

by the formula

$$h(b,x) = x_1 s_{m+1}(b) + \ldots + x_{n-m} s_n(b).$$

□

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gives an isomorphism of $\tau \oplus \nu$ with $S^{n-1} \times \mathbb{R}^n$. Since the normal bundle ν is trivial, this shows that τ is stably trivial.

But there are also examples where both bundle are nontrivial whereas their Whitney sum is trivial.

Example 5.10. Let γ_n^1 be the canonical line bundle on $\mathbb{R}\mathrm{P}^n$. Then the map $(\ell, v, w) \mapsto (\ell, v + w)$ for $v \in \ell$ and $w \perp \ell$ defines an isomorphism $\gamma_n^1 \oplus (\gamma_n^1)^{\perp} \cong \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$.

Example 5.11. Specializing the previous example to the case $n = 1$, we see that

$$\gamma_1^1 \oplus (\gamma_1^1)^{\perp} \cong \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^2 \cong S^1 \times \mathbb{R}^2.$$

The map that rotates a vector by 90 degrees defines an isomorphism between $(\gamma_1^1)^{\perp}$ and γ_1^1. Since γ_1^1 is isomorphic to the Möbius bundle over S^1, this shows that the direct sum of the Möbius bundle with itself is the trivial bundle.

5.4. **Oriented bundles.** We start with a first working definition of orientation of a vector bundle. Later we will discuss orientations in a more general context and relate it elements in the cohomology groups of the total space.

Recall that an *orientation* of a real vector space V of dimension $n > 0$ is an equivalence class of bases, where two ordered bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are said to be equivalent if and only if the matrix (a_{ij}) defined by the equation

$$v'_i = \sum a_{ij} v_j$$

has positive determinant. Evidently every such vector space V has precisely two distinct orientations.

Example 5.12. The vector space $\mathbb{R}^n$ has a canonical orientation corresponding to its canonical ordered basis.

Definition 5.13. Let ξ be a real vector bundle given by the map $\pi\colon E \to B$. An *orientation* of ξ is a function assigning an orientation to each fiber in such a way that near each point of B there is a local trivialization $h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ carrying the canonical orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$ to the orientations of the fibers in $\pi^{-1}(U)$.

An oriented vector bundle ξ is a real vector bundle together with a choice of orientation.

Note: Not all bundles can be oriented.

Example 5.14. a) Every trivial bundle is orientable. Hence the existence of an orientation is a necessary condition for triviality.

b) The Möbius bundle is not orientable.

We will see in the next lecture that the Stiefel-Whitney class measures exactly if a bundle is orientable or not.

6. Stiefel–Whitney classes and embedding problems

This was a guest lecture by Mike Hopkins. Unfortunately, there are no notes available.

7. Stiefel–Whitney classes of projective spaces

Our next goal is to apply Stiefel–Whitney classes to prove the following important result by Stiefel.

7.1. **Division algebras and projective spaces.**

Theorem 7.1. *Suppose that there is a structure of a division algebra on* $\mathbb{R}^n$*. Then the projective space* $\mathbb{P}^{n-1}$ *is parallelizable. In particular,* n *must be a power of* 2*.*

Remark 7.2. In fact, we know that there is the much stronger result that a division algebra structure exists on $\mathbb{R}^n$ if and only if $n = 1, 2, 4, 8$. But to prove this final result we need stronger techniques. So for a moment let's be modest and see how the methods we know so far lead to a proof of this algebraic result.

7.2. **Stiefel–Whitney classes of projective spaces.**

Example 7.3. Stiefel–Whitney classes are not fine enough to decide if the tangent bundle of a sphere is trivial or not. For the tangent bundle of a sphere is stably trivial, hence $w(S^n) = w(\tau_{S^n}) = 1$.

Lemma 7.4. *The total Stiefel–Whitney class of the canonical bundle* γ_n^1 *over* $\mathbb{P}^n$ *is given by*

$$w(\gamma_n^1) = 1 + a$$

where a *denotes the nonzero element of* $H^1(\mathbb{P}^n; \mathbb{Z}/2)$*.*

Proof. The standard inclusion $j\colon \mathbb{P}^1 \to \mathbb{P}^n$ is clearly covered by a bundle map from γ_1^1 to γ_n^1. Therefore

$$j^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0.$$

Hence $w_1(\gamma_n^1)$ cannot be zero, hence it must be equal to a. Since γ_n^1 is a line bundle, the first axiom for Stiefel–Whitney classes tells us that the higher classes must be zero. □

Example 7.5. The canonical line bundle γ_n^1 over $\mathbb{P}^n$ is contained as a sub-bundle in the trivial bundle ϵ^{n+1}. Let $\gamma^\perp$ denote the orthogonal complement of γ_n^1 in ϵ^{n+1}. The total space $E(\gamma^\perp)$ consists of all pairs

$$(\{\pm x\}, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1}$$

with v orthogonal to x. Claim:

$$w(\gamma^\perp) = 1 + a + a^2 + \ldots + a^n.$$

For: Since $\gamma_n^1 \oplus \gamma^\perp$ is trivial we have

$$w(\gamma^\perp) = \bar{w}(\gamma_n^1) = (1+a)^{-1} = 1 + a + a^2 + \ldots + a^n.$$

In particular, we see that it is possible that all of the n Stiefel–Whitney classes of an $\mathbb{R}^n$-bundle can be non-zero.

Lemma 7.6. *The tangent bundle τ of $\mathbb{P}^n$ is isomorphic to* $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$.

Proof. Let L be a line through the origin in $\mathbb{R}^{n+1}$, intersecting S^n in the points $\pm x$, and let $L^\perp \subset \mathbb{R}^{n+1}$ be the complementary n-plane. Let $f\colon S^n \to \mathbb{P}^n$ denote the canonical map $f(x) = \{\pm x\}$. Note that the two tangent vectors (x,v) and $(-x,-v)$ in DS^n both have the same image under the map

$$Df\colon DS^n \to D\mathbb{P}^n$$

which is induced by f. Thus the tangent manifold $D\mathbb{P}^n$ can be identified with the set of pairs $\{(x,v),(-x,-v)\}$ satisfying

$$x \cdot x = 1,\ v \cdot v = 0.$$

But each such pair determines, and is determined by, a linear mapping

$$\ell\colon L \to L^\perp,$$

where

$$\ell(x) = v.$$

Thus the tangent space of $\mathbb{P}^n$ at $\{\pm x\}$ is canonically isomorphic to the vector space $\mathrm{Hom}(L,L^\perp)$. It follows that the tangent vector bundle $\tau = \tau_{\mathbb{P}^n}$ is isomorphic to the bundle $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$. □

Let us compute the total Stiefel–Whitney class $w(\mathbb{P}^n)$. We cannot use the previous formula for τ, since we do not a formula that relates the Stiefel–Whitney classes of $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$, γ_n^1, and $\gamma^\perp$. Instead we do the following.

Theorem 7.7. *the Whitney sum $\tau \oplus \epsilon^1$ is isomorphic the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \gamma_n^1 \oplus \ldots \oplus \gamma_n^1$. Hence the total Stiefel–Whitney class of $\mathbb{P}^n$ is given by*

$$w(\mathbb{P}^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \ldots + \binom{n+1}{n} a^n.$$

Proof. The bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is trivial since it is a line bundle with a canonical nowhere zero section. Therefore

$$\tau \oplus \epsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1).$$

But the latter is isomorphic to

$$\mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}),$$

and therefore it is isomorphic to the $(n+1)$-fold sum

$$\mathrm{Hom}(\gamma_n^1, \epsilon^1 \oplus \ldots \oplus \epsilon^1) \cong \mathrm{Hom}(\gamma_n^1, \epsilon^1) \oplus \ldots \oplus \mathrm{Hom}(\gamma_n^1, \epsilon^1).$$

But the bundle $\mathrm{Hom}(\gamma_n^1, \epsilon^1)$ is isomorphic to γ_n^1, since γ_n^1 has a Euclidean metric. This proves that

$$\tau \oplus \epsilon^1 \cong \gamma_n^1 \oplus \ldots \oplus \gamma_n^1.$$

The Whitney product formula implies that $w(\tau) = w(\tau \oplus \epsilon^1)$ is equal to

$$w(\gamma_n^1) \ldots w(\gamma_n^1) = (1+a)^{n+1}.$$

The binomial formula now completes the proof. □

Corollary 7.8. *The class $w(\mathbb{P}^n)$ is equal to 1 if and only if $n+1$ is a power of 2. Thus the only projective spaces which can be parallelizable are $\mathbb{P}^1, \mathbb{P}^3, \mathbb{P}^7, \mathbb{P}^{15}, \ldots$.*

Proof. The identity $(a+b)^2 = a^2 + b^2$ modulo 2 implies that

$$(1+a)^{2^r} = 1 + a^{2^r}.$$

Therefore if $n+1 = 2^r$ then

$$w(\mathbb{P}^n) = (1+a)^{n+1} = 1 + a^{n+1} = 1.$$

Conversely if $n+1 = 2^r m$ with m odd, m>1, then

$$\begin{aligned} w(\mathbb{P}^n) &= (1+a)^{n+1} = (1+a^{2^r})^m \\ &= 1 + ma^{2^r} + \tfrac{m(m-1}{2} a^{2\cdot 2^r} + \ldots \neq 1, \end{aligned}$$

since $2^r < n+1$. □

7.3. **Proof of Stiefel's theorem.** Assume there is a bilinear product operation

$$p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

without zero divisors.

Let $b_1, \ldots, b_n$ be the standard basis for the vector space $\mathbb{R}^n$. The correspondence

$$y \mapsto p(y,b_1)$$

defines an isomorphism of $\mathbb{R}^n$ onto itself, since p has no zero divisors. Hence the formula

$$v_i(p(y,b_1)) = p(y,b_i)$$

defines a linear transformation

$$v_i \colon \mathbb{R}^n \to \mathbb{R}^n.$$

Note that we have $v_1(x) = x$, since $v_1(p(y,b_1)) = p(y,b_1)$ by definition. Moreover, for $x \neq 0$, the vectors $v_1(x), \ldots, v_n(x)$ are linearly independent. For if there was a nontrivial relation, for some $y \in \mathbb{R}^n$ with $x = p(y,b_1)$,

$$0 = \sum_i \lambda_i v_i(x) = \sum_i \lambda_i p(y,b_i) = p(y, \sum_i \lambda_i b_i)$$

this implied

$$0 = \sum_i \lambda_i b_i$$

which implies $\lambda_i = 0$ for all i.

Now let L be a line through the origin. Each v_i defines a linear transformation

$$\bar{v}_i \colon L \to L^{\perp}$$

as follows. For $x \in L$, let $\bar{v}_i(x)$ denote the image of $v_i(x)$ under the orthogonal projection

$$\mathbb{R}^n \to L^{\perp}.$$

Since $v_1(x) = x$, we have $\bar{v}_1 = 0$. But the $\bar{v}_2, \ldots, \bar{v}_n$ are everywhere linearly independent, since the $v_2, \ldots, v_n$ are everywhere linearly independent. Hence the $v_2, \ldots, v_n$ give rise to $n-1$ linearly independent sections of the bundle

$$\mathrm{Hom}(\gamma_n^1, \gamma^{\perp}).$$

Since this bundle is isomorphic the tangent bundle $\tau_{\mathbb{P}^{n-1}}$ of $\mathbb{P}^{n-1}$, we see that $\tau_{\mathbb{P}^{n-1}}$ is trivial. This completes the proof of Theorem 7.1.

8. Existence and uniqueness of Stiefel–Whitney classes I

Before we show that Stiefel–Whitney classes with the described properties actually exist we are going to see another interesting application of Stiefel–Whitney classes.

8.1. **Immersions of projective spaces into Euclidean space.** Stiefel-Whitney classes also help us decide whether a manifold can be immersed into a Euclidean space. For if an n-dimensional manifold M can be immersed into $\mathbb{R}^{n+k}$ then

$$1 = w(\tau_{\mathbb{R}^{n+k}}) = w(\nu \oplus \tau_M)$$

where ν denotes the normal bundle of the embedding $M \subset \mathbb{R}^{n+k}$. Hence by the Whitney product formula

$$w_i(\nu) = \bar{w}_i(M)$$

where $\bar{w}_i(M)$ denotes the ith component of the multiplicative inverse of the total Stiefel–Whitney class $w(M)$. Since ν is a k-dimensional bundle, this shows

$$\bar{w}_i(M) = 0 \text{ for } i > k.$$

Example 8.1. A typical example is the real projective space $\mathbb{P}^9$. By our calculations in the previous lecture we know

$$w(\mathbb{P}^9) = (1+a)^{10} = 1 + \sum_{i=1}^{9} \binom{10}{i} a^i = 1 + a^2 + a^8$$

since all other terms have an even coefficient. As a multiplicative inverse we get

$$\bar{w}(\mathbb{P}^9) = 1 + a^2 + a^4 + a^6,$$

for

$$\begin{aligned} & (1+a^2+a^8)(1+a^2+a^4+a^6) \\ = \ & 1 + a^2 + a^4 + a^6 + a^2 + a^4 + a^6 + a^8 + a^8 + a^{10} + a^{12} + a^{14} \\ = \ & 1 + 2a^2 + 2a^4 + 2a^6 + 2a^8 \\ = \ & 1. \end{aligned}$$

Since $\bar{w}_6(\mathbb{P}^9) \neq 0$, k must be at least 6 if $\mathbb{P}^9$ can be immersed into $\mathbb{R}^{9+k}$.

If $n = 2^r$ is a power of 2, then

$$w(\mathbb{P}^n) = (1+a)^{2^r+1} = (1+a^n)(1+a) = 1 + a + a^n$$

and

$$\bar{w}(\mathbb{P}^n) = 1 + a + a^2 + \ldots + a^{n-1}$$

since

$$\begin{aligned} & (1+a+a^{2^r})(1+a+\ldots+a^{n-1}) \\ = \ & 1+a+\ldots+a^{n-1}+a+a^2+\ldots+a^n+a^n \\ = \ & 1+2(a+a^2+\ldots+a^n) \\ = \ & 1. \end{aligned}$$

Together with the previous argument we get the following classical result.

Theorem 8.2. *If $\mathbb{P}^{2^r}$ can be immersed in $\mathbb{R}^{2^r+k}$, then k must be at least 2^r-1.*

Example 8.3. Since the theorem tells us that $\mathbb{P}^8$ cannot be immersed in $\mathbb{R}^{14}$, it follows that $\mathbb{P}^9$ cannot be immersed in $\mathbb{R}^{14}$ either. This gives another proof that the minimal dimension of a Euclidean space in which $\mathbb{P}^9$ can be immersed is 15.

8.2. **Existence of Stiefel-Whitney classes.** We still need to show that there cohomology classes that satisfy the axioms of Stiefel-Whitney classes.

Theorem 8.4. *There is a unique sequence of functions $w_1, w_2, \ldots$ assigning to each real vector bundle $E \to B$ over a a space B a class $w_i(E) \in H^i(B;\mathbb{Z}/2)$, depending only on the isomorphism type of E, such that*
a) $w_i(f^*E) = f^*(w_i(E))$ for a pullback along a map $f\colon B' \to B$ which is covered by a bundle map.
b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$ where $w = 1 + w_1 + w_2 + \ldots \in H^*(B;\mathbb{Z}/2)$.
c) $w_i(E) = 0$ if $i > \dim E$.
d) For the canonical line bundle γ_1^1 on $\mathbb{P}^1$, $w_1(\gamma_1^1)$ is non-zero.

There are different methods to prove this theorem. We will prove it using the following fundamental result of Leray and Hirsch on the cohomology of a fiber bundle. Roughly speaking, a fiber bundle is the same thing as a vector bundle except that we replace $\mathbb{R}^n$ by any topological space F.

Let $p\colon E \to B$ be a fiber bundle with fiber F. Then we can make $H^*(E;\mathbb{Z}/2)$ into a module over the ring $H^*(B;\mathbb{Z}/2)$ by setting $\alpha\beta = p^*(\alpha)\beta$ for $\alpha \in H^*(B;\mathbb{Z}/2)$ and $\beta \in H^*(E;\mathbb{Z}/2)$. The Leray–Hirsch theorem then tells us that $H^*(E;\mathbb{Z}/2)$ is a free $H^*(B;\mathbb{Z}/2)$-module provided that for each fiber F the inclusion $\iota\colon F \hookrightarrow E$ induces a surjection on $H^*(F;\mathbb{Z}/2)$ and $H^n(F;\mathbb{Z}/2)$ is a finite dimensional $\mathbb{Z}/2$-vector space for each n. A basis for $H^*(E;\mathbb{Z}/2)$ as a $H^*(B;\mathbb{Z}/2)$-module can be chosen as any set of elements in $H^*(E;\mathbb{Z}/2)$ that map to a basis in $H^*(F;\mathbb{Z}/2)$ under ι^*.

The precise statement of the Leray–Hirsch theorem is:

Theorem 8.5. *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative ring R,*
a) $H^n(F;R)$ is a finitely generated free R-module for each n,
b) and there exist classes $c_j \in H^{k_j}(E;R)$ whose restrictions $\iota^*(c_j)$ form a basis for $H^*(F;R)$ in each fiber F.
Then the map $\varphi\colon H^*(B;R)\otimes_R H^*(F;R) \to H^*(E;R)$, $\sum_{ij} b_i \otimes \iota^*(x_j) \mapsto \sum_{ij} p^*(b_i)x_j$, is an isomorphism.

Now let us prove Theorem 9.1. For simplicity, we will assume that the base base is paracompact.

Let ξ be a vector bundle of dimension n given by the map $\pi\colon E \to B$. It comes along with a projective bundle $\mathbb{P}(\xi)$ given by the induced map $\mathbb{P}(\pi)\colon \mathbb{P}(E) \to B$. It is a fiber bundle whose fiber at b in B are the spaces of all lines through the origin in the fiber $E_b(\xi)$. The map $\mathbb{P}(\pi)$ is the natural projection sending each line in $\pi^{-1}(b)$ to b. We topologize $\mathbb{P}(E)$ as a quotient of the complement of the zero section of E modulo scalar multiplication in each fiber. Over a neighborhood U in B where E is a product $U \times \mathbb{R}^n$, this quotient is $U \times \mathbb{P}^{n-1}$. Hence $\mathbb{P}(\xi)$ is a fiber bundle over B with fiber $\mathbb{P}^{n-1}$.

Now we would like to apply the Leray–Hirsch theorem to the fiber bundle $\mathbb{P}(\xi)$. Therefore we need classes $x_i \in H^i(\mathbb{P}(E);\mathbb{Z}/2)$ restricting to generators of $H^i(\mathbb{P}^{n-1};\mathbb{Z}/2)$ in each fiber $\mathbb{P}^{n-1}$ for $i = 0,\dots,n-1$.

We will use the following lemma.

Lemma 8.6. *There is a map $g\colon E \to \mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$ that is a linear injection on each fiber. Any two such maps are homotopic through maps that are linear injections on fibers.*

Proof. Since B is paracompact there is a countable open cover U_j of B such that E is trivial over each U_j and there is a partition of unity $\{\varphi_j\}$ with φ_j supported on U_j. Let $g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n$ be the composition of a trivialization $\pi^{-1}(U_j) \to U_j \times \mathbb{R}^n$ with the projection onto $\mathbb{R}^n$. The map

$$(\varphi_j\pi)g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n,\ v \mapsto \varphi_j(\pi(v))g_j(v)$$

extends to a map $E \to \mathbb{R}^n$ that is zero outside $\pi_{-1}(U_j)$. Near each point of B only finitely many φ_j's are nonzero, and at least one φ_j is nonzero. Hence these extended maps $(\varphi_j\pi)g_j$ are the coordinates of a map $g\colon E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ that is a linear injection on each fiber.

Now let g_0 and g_1 be two such maps that are linear injections on fibers. Then let L_t be the homotopy

$$L_t \colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ L_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots).$$

For each t, this is a linear map whose kernel is easily computed to be 0. Hence L_t is injective. Composing L_t with g_0 moves the image of g_0 into the odd-numbered coordinates. Similarly, we can move the image of g_1 into the even-numbered coordinates. By abuse of notation we denote the resulting shifted maps still by g_0 and g_1 respectively. Then we set

$$g_t = (1-t)g_0 + tg_1.$$

This is a linear map which is injective on fibers for each t since g_0 and g_1 are linear and injective on fibers. □

Given the linear injection g of the lemma, we can projectivize it by deleting zero vectors and then take the quotient by scalar multiplication. This gives us a map $\mathbb{P}(g) \colon \mathbb{P}(E) \to \mathbb{P}^\infty$. Let y be a generator of $H^1(\mathbb{P}^\infty; \mathbb{Z}/2)$ and let $x = \mathbb{P}(g)^*(y) \in H^1(\mathbb{P}(E); \mathbb{Z}/2)$. Then the powers $x_i := x^i \in H^i(\mathbb{P}(E); \mathbb{Z}/2)$ for $i = 0, \ldots, n-1$ are the desired classes since a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ induces an embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ for which y pulls back to a generator of $H^1(\mathbb{P}^{n-1}; \mathbb{Z}/2)$ (because the classes are nonzero).

Note that the classes x^i do not depend on the choice of g. For any two linear injections $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ of different fibers of $\mathbb{P}(E)$ are all homotopic. The second assertion of the lemma then implies the claim.

Hence, by the Leray–Hirsch theorem, $H^*(\mathbb{P}(E); \mathbb{Z}/2)$ is a free $H^*(B; \mathbb{Z}/2)$-module with basis $1, x, \ldots, x^{n-1}$. Consequently, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B; \mathbb{Z}/2)$. Thus there is a unique relation of the form

$$x^n + w_1(E)x^{n-1} + \ldots + w_n(E) = 0$$

for certain classes $w_i(E) \in H^i(B; \mathbb{Z}/2)$. Together with the convention $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$ this is our definition of the Stiefel–Whitney classes of E. It remains to show that these classes satisfy the desired properties.

9. Existence and uniqueness of Stiefel–Whitney classes II

We continue the proof of the following theorem that shows that there exist unique Stiefel–Whitney classes.

Theorem 9.1. *There is a unique sequence of functions $w_1, w_2, \ldots$ assigning to each real vector bundle $E \to B$ over a a space B a class $w_i(E) \in H^i(B;\mathbb{Z}/2)$, depending only on the isomorphism type of E, such that*
a) $w_i(f^*E) = f^*(w_i(E))$ for a pullback along a map $f\colon B' \to B$ which is covered by a bundle map.
b) $w(E_1 \oplus E_2) = w(E_1)w(E_2)$ where $w = 1 + w_1 + w_2 + \ldots \in H^*(B;\mathbb{Z}/2)$.
c) $w_i(E) = 0$ if $i > \dim E$.
d) For the canonical line bundle γ_1^1 on $\mathbb{P}^1$, $w_1(\gamma_1^1)$ is non-zero.

9.1. **Existence of Stiefel–Whitney classes.** In the previous lecture we defined the Stiefel–Whitney classes $w_i(E)$ for any vector bundle $\pi\colon E \to B$. Recall that for simplicity we assume that the base space B is paracompact. The idea was the following.

Our bundle induces a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber. We can projectivize it by deleting zero vectors and then take the quotient by scalar multiplication. This gives us a map $\mathbb{P}(g)\colon \mathbb{P}(E) \to \mathbb{P}^\infty$. Let y be a generator of $H^1(\mathbb{P}^\infty;\mathbb{Z}/2)$ and let $x = \mathbb{P}(g)^*(y) \in H^1(\mathbb{P}(E);\mathbb{Z}/2)$. Then the powers $x_i := x^i \in H^i(\mathbb{P}(E);\mathbb{Z}/2)$ for $i = 0, \ldots, n-1$ are the desired classes since a linear injection $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ induces an embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ for which y pulls back to a generator of $H^1(\mathbb{P}^{n-1};\mathbb{Z}/2)$ (because the classes are nonzero).

Note that the classes x^i do not depend on the choice of g. For any two linear injections $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^\infty$ of different fibers of $\mathbb{P}(E)$ are all homotopic. The second assertion of the lemma then implies the claim.

Hence, by the Leray-Hirsch theorem, $H^*(\mathbb{P}(E);\mathbb{Z}/2)$ is a free $H^*(B;\mathbb{Z}/2)$-module with basis $1, x, \ldots, x^{n-1}$. Consequently, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B;\mathbb{Z}/2)$. Thus there is a unique relation of the form

$$x^n + w_1(E)x^{n-1} + \ldots + w_n(E) = 0$$

for certain classes $w_i(E) \in H^i(B;\mathbb{Z}/2)$. Together with the convention $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$ this is our definition of the Stiefel-Whitney classes of E. It remains to show that these classes satisfy the desired properties.

a) Consider a pullback bundle $f^*E = E'$:

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

If $g\colon E \to \mathbb{R}^\infty$ is a map that is a linear injection on fibers then so is gf'. It follows that $\mathbb{P}(f')^*$ takes the canonical class $x = x(E)$ in $H^1(\mathbb{P}(E); \mathbb{Z}/2)$ to the canonical class $x(E')$ in $H^1(\mathbb{P}(E'); \mathbb{Z}/2)$. Then

$$\begin{array}{rcl} \mathbb{P}(f')^*(\sum_i \mathbb{P}(\pi)^*(w_i(E)) \cdot x(E)^{n-i}) & = & \sum_i [\mathbb{P}(f')^* \circ \mathbb{P}(\pi)^*(w_i(E))] \cdot [\mathbb{P}(f')^*(x(E)^{n-i})] \\ & = & \sum_i \mathbb{P}(\pi')^* \circ f^*(w_i(E) \cdot x(E'))^{n-i} \end{array}$$

in $H^*(E'; \mathbb{Z}/2)$. This shows that the relation

$$x(E)^n + w_1(E)x(E)^{n-1} + \ldots + w_n(E) = 0 \text{ defining } w_i(E)$$

pulls back to the relation

$$x(E')^n + f^*w_1(E)x(E')^{n-1} + \ldots + f^*w_n(E) = 0 \text{ defining } w_i(E').$$

By the uniqueness of this relation in the free $H^*(B; \mathbb{Z}/2)$-module $H^*(E; \mathbb{Z}/2)$, we get $w_i(E') = f^*(w_i(E))$.

b) The inclusions of E_1 and E_2 into $E_1 \oplus E_2$ give inclusions of $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ into $\mathbb{P}(E_1 \oplus E_2)$ with $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset$. Let $U_1 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_1)$ and $U_2 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_2)$. These are open sets in $\mathbb{P}(E_1 \oplus E_2)$ which cover $\mathbb{P}(E_1 \oplus E_2)$ and that deformation retract onto $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ respectively. This means that the inclusions $\mathbb{P}(E_1) \hookrightarrow U_2$ and $\mathbb{P}(E_2) \hookrightarrow U_1$ are homotopy equivalences.

A map $g\colon E_1 \oplus E_2 \to \mathbb{R}^\infty$ which is a linear injection on fibers restricts to such a map on E_1 and E_2. By the way we constructed the canonical classes, this implies that the canonical class $x \in H^1(\mathbb{P}(E_1 \oplus E_2; \mathbb{Z}/2)$ for $E_1 \oplus E_2$ restricts to the canonical classes for E_1 and E_2.

If E_1 and E_2 have dimensions m and n, we consider the classes

$$\omega_1 = \sum_j w_j(E_1)x^{m-j} \text{ and } \omega_2 = \sum_j w_j(E_2)x^{n-j} \text{ in } H^*(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}/2).$$

Their cup product is

$$\omega_1 \cdot \omega_2 = \sum_j [\sum_{r+s=j} w_r(E_1)w_r(E_2)]x^{m+n-j}.$$

By the definition of the classes $w_j(E_1)$, the class ω_1 restricts to zero in $H^m(\mathbb{P}(E_1); \mathbb{Z}/2)$. Hence ω_1 pulls back to a class in the relative group

$$H^m(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1); \mathbb{Z}/2) \cong H^m(\mathbb{P}(E_1 \oplus E_2), U_2; \mathbb{Z}/2).$$

and ω_2 pulls back to a class in the relative group

$$H^n(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_2); \mathbb{Z}/2) \cong H^n(\mathbb{P}(E_1 \oplus E_2), U_1; \mathbb{Z}/2).$$

The following commutative diagram then shows that $\omega_1 \cdot \omega_2 = 0$:

$$\begin{array}{ccc} H^m(\mathbb{P}(E_1 \oplus E_2), U_2; \mathbb{Z}/2) \times H^n(\mathbb{P}(E_1 \oplus E_2), U_1; \mathbb{Z}/2) & \longrightarrow & H^{m+n}(\mathbb{P}(E_1 \oplus E_2), U_1 \cup U_2; \mathbb{Z}/2) = 0 \\ \downarrow & & \downarrow \\ H^m(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}/2) \times H^n(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}/2) & \longrightarrow & H^{m+n}(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}/2). \end{array}$$

This shows that

$$\omega_1 \cdot \omega_2 = \sum_j [\sum_{r+s=j} w_r(E_1) w_r(E_2)] x^{m+n-j} = 0$$

is the defining relation for the Stiefel–Whitney classes of $E_1 \oplus E_2$. Thus

$$w_j(E_1 \oplus E_2) = \sum_{r+s=j} w_r(E_1) w_r(E_2).$$

c) holds by definition.

d) Recall that the canonical line bundle γ^1 on $\mathbb{P}^\infty$ is given by

$$E(\gamma^1) = \{(\ell, v) \in \mathbb{P}^\infty \times \mathbb{R}^\infty | v \in \ell\}.$$

The map $\mathbb{P}(\pi)$ is the identity in this case, i.e. γ^1 is equal to its own projective bundle. The map $g \colon E \to \mathbb{R}^\infty$ which is a linear injection on fibers can be taken to be

$$g(\ell, v) = v.$$

So $\mathbb{P}(g)$ is also the identity and $x(E)$ is a generator of $H^1(\mathbb{P}^\infty; \mathbb{Z}/2)$ and restricts to the generator in $H^1(\mathbb{P}^1; \mathbb{Z}/2)$. This proves the existence of Stiefel–Whitney classes.

9.2. **Uniqueness.** To show the uniqueness we will use an important property of vector bundles, the *splitting principle*:

Proposition 9.2. *For each vector bundle $\pi \colon E \to B$ there is a space $F(E)$ and a map $P \colon F(E) \to B$ such that the pullback $p^*(E) \to F(E)$ splits as a direct sum of line bundles, and $p^* \colon H^*(B; \mathbb{Z}/2) \to H^*(F(E); \mathbb{Z}/2)$ is injective.*

Now we can finish the proof of Theorem 9.1 and show the uniqueness of Stiefel–Whitney classes. Property d) determines $w_1(\gamma^1)$ for the canonical line bundle

$\gamma^1 \to \mathbb{P}^\infty$. Property c) then determines all the $w_i(\gamma^1)$'s. We will now use the following property of the line bundle γ^1.

Remark 9.3. The canonical line bundle γ^1 on $\mathbb{P}^\infty$ is the universal line bundle in the following sense. Given a line bundle ξ, then there is a bundle map $f\colon \xi \to \gamma^1$ which is unique up to homotopy. For let ξ be given by a map $\pi\colon E \to B$. We have seen in the previous lecture that we can find a map $g\colon E \to \mathbb{R}^\infty$ that is linear and injective on fibers. Then we can define f by

$$f(e) = (g(\text{fiber through } e), g(e)) \in \gamma^1.$$

Using the universality of γ^1, we see that property a) therefore determines the classes w_i for all line bundles. Property b) extends this to sums of line bundles. Finally, the splitting principle implies that the $w_i's$ are determined for all bundles.

10. Splitting principle and the projective bundle formula

There are two leftovers from the proof of the existence and uniqueness of Stiefel–Whitney classes. One is the splitting principle, the other one is the Leray–Hirsch theorem.

10.1. **The splitting principle.**

Proposition 10.1. *For each vector bundle $\pi\colon E \to B$ there is a space $F(E)$ and a map $p\colon F(E) \to B$ such that the pullback $p^*(E) \to F(E)$ splits as a direct sum of line bundles, and $p^*\colon H^*(B;\mathbb{Z}/2) \to H^*(F(E);\mathbb{Z}/2)$ is injective.*

Proof. Consider the pullback $\mathbb{P}(\pi)^*(E)$ of E via the map $\mathbb{P}(\pi)\colon \mathbb{P}(E) \to B$. This pullback contains a natural one-dimensional sub-bundle

$$L = \{(\ell,v) \in \mathbb{P}(E) \times E | v \in \ell\}.$$

Assuming B is paracompact (although this holds for any B) we can equip E with an inner product. This inner product pulls back to an inner product on $\mathbb{P}(\pi)^*(E)$. Hence we get a splitting of the pullback as a sum $L \oplus L^\perp$. The orthogonal bundle $L^\perp$ now has dimension less than E. By the Leray–Hirsch theorem we know $H^*(\mathbb{P}(E);\mathbb{Z}/2)$ is the free $H^*(B;\mathbb{Z}/2)$-module with basis $1, x, \dots, x^{n-1}$. In particular, the induced map

$$H^*(B;\mathbb{Z}/2) \to H^*(\mathbb{P}(E);\mathbb{Z}/2)$$

is injective since one of the basis elements is 1.
Now we can repeat this construction for the bundle $L^\perp \to \mathbb{P}(E)$ instead of $E \to B$. After finitely many steps we obtain the desired result. □

Remark 10.2. We can describe $F(E)$ as follows. The complement $L^\perp$ consist of pairs $(\ell,v) \in \mathbb{P}(E) \times E$ with $v \perp \ell$. At the next stage we construct $\mathbb{P}(L^\perp)$, whose points are pairs (ℓ,ℓ') where ℓ and ℓ' are orthogonal lines in E. Continuing this way, we see that the final space $F(E)$ is the space of all orthogonal splittings $\ell_1 \oplus \dots \oplus \ell_n$ of fibers of E as sums of lines, and the vector bundle over $F(E)$ consists of all n-tuples of vectors in these lines.

In the previous proof we used the following result.

Proposition 10.3. *Let B be a paracompact space and ξ a real vector bundle given by the map $\pi\colon E \to B$. Then ξ can be given the structure of a Euclidean vector bundle.*

Proof. See problem set 1. □

10.2. **The Leray–Hirsch theorem.** The precise statement of the Leray-Hirsch theorem is:

Theorem 10.4. *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fiber bundle such that for a principal ideal ring R,*
a) $H^n(F;R)$ is a finitely generated free R-module for each n,
b) and there exist classes $c_j \in H^{k_j}(E;R)$ for $j = 1, \ldots, r$ whose restrictions $\iota^*(c_j)$ form a basis for the R-module $\oplus_n H^n(F;R)$ in each fiber F.
Then the map $\varphi\colon H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$, $\sum_{ij} b_i \otimes \iota^*(c_j) \mapsto \sum_{ij} p^*(b_i)c_j$, is an isomorphism.

Remark 10.5. 1. Note that the theorem makes only an assertion on the structure of $H^*(E;R)$ as an $H^*(B;R)$-module. It does not specify the ring structure of $H^*(E;R)$. In fact, there are examples where the map

$$\varphi\colon H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$$

is not a ring isomorphism.

2. An example of a fiber bundle where the assertion of the theorem does not hold is the Hopf bundle

$$S^1 \to S^3 \xrightarrow{f} S^2.$$

(Recall that f can be defined as $f\colon S^3 \to \mathbb{CP}^1 = S^2$, viewing S^3 as the unit sphere in the complex plane $\mathbb{C}^2$. Such an f is the attaching map in the complex projective plane $\mathbb{CP}^2 = S^2 \cup_f e^4$ where e^4 is a disk of dimension 4.)

We know that $H^*(S^3;R)$ is not isomorphic to $H^*(S^2;R) \otimes_R H^*(S^1;R)$. For we have

$$H^1(S^3;R) = 0 \text{ but } H^0(S^2;R) \otimes_R H^1(S^1;R) \cong R.$$

The assumptions of the theorem require that the map $\iota^*\colon H^*(E;R) \to H^*(F;R)$ is surjective in each degree. This is obviously not the case for the Hopf bundle.

Sketch of a proof of Theorem 10.4 for compact base spaces:

Throughout the proof we write $H^*(X)$ for $H^*(X;R)$. We only sketch a proof for the case that B is compact, though the theorem holds for arbitrary base spaces.

Let Ube an open subset of B such that there is a homeomorphism

$$h\colon E_U := \pi^{-1}(U) \to U \times F.$$

Let $j_U\colon E_U \hookrightarrow E$ be the natural inclusion and π_U be the restriction of π to U. Then the Künneth Theorem says that the map $\pi_{U*}\colon H^*(U) \to H^*(E_U)$ is injective and the elements $j_U^*(c_1), \ldots, j_U^*(c_r)$ form a basis of the $H^*(U)$-module $H^*(E_U)$.

Now assume that the theorem is true over the open subsets U, V and $U \cap V$. We want to show that it is also true over $U \cup V$. Therefore we introduce two functors $K^n(W)$ and $L^n(W)$ on the open subsets W of B as follows. Let t_j be an indeterminant of degree k_j. (The t_j have no real meaning, they are just useful to define something else.) We set

$$K^n(W) := \sum_{j=1}^{r} H^{n-k_j}(W)t_j, \text{ and } L^n(W) := H^n(E_W).$$

For every W we have the homomorphism

$$\theta_W \colon K^n(W) \to L^n(W), \ \sum_j x_j t_j \mapsto \sum_j \pi^*(x_j)c_j.$$

Then we convince ourselves that the theorem is true over W if and only if θ_W is an isomorphism.

The functor $W \mapsto L^n(W)$ is just the restriction of a functor which satisfies the Mayer–Vietoris property. The functor $W \mapsto K^n(W)$ is a direct sum of functors which satisfy the Mayer–Vietoris property. Hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
K^{n-1}(U) \oplus K^{n-1}(V) & \longrightarrow & K^{n-1}(U \cap V) & \longrightarrow & K^n(U \cup V) & \longrightarrow & K^n(U) \oplus K^n(V) & \longrightarrow & K^n(U \cap V) \\
\downarrow & & \downarrow & & \downarrow \theta_{U \cup V} & & \downarrow & & \downarrow \\
L^{n-1}(U) \oplus L^{n-1}(V) & \longrightarrow & L^{n-1}(U \cap V) & \longrightarrow & L^n(U \cup V) & \longrightarrow & L^n(U) \oplus L^n(V) & \longrightarrow & L^n(U \cap V)
\end{array}$$

By our assumption the theorem is true for U, V and $U \cap V$ and hence the four unlabelled vertical maps are isomorphisms. By the 5-Lemma, the map $\theta_{U \cup V}$ is thus an isomorphism too. Hence the theorem also holds over $U \cup V$.

Now it remains to cover B by finitely many open sets $B = U_1 \cup \ldots \cup U_n$ such that our bundle becomes trivial over each U_i. This completes the proof for a compact base space.

More sophisticated arguments using the Serre spectral sequence associated to the fibration sequence $F \xrightarrow{\iota} E \xrightarrow{p} B$ also prove the general case. A more elementary proof of the general statement can be found in Hatcher's book (Theorem 4D.1).

11. The Grassmannian manifold and the universal bundle

11.1. **Representability of** $\mathrm{Vect}^k(B)$**.** In the previous lecture we used the fact that the canonical line bundle γ^1 over the (infinite) real projective space is universal among all line bundles in the sense that given a line bundle ξ there is a bundle map $\xi \to \gamma^1$ which is unique up to homotopy. This bundle map comes equipped with a homotopy class of maps $B \to \mathbb{P}^\infty$ where B denotes the base space of ξ. In fact, there is a bijection

$$\mathrm{Vect}^1(B) \cong [B, \mathbb{P}^\infty]$$

between the set of isomorphism classes of real line bundles over B and the set of homotopy classes of maps $B \to \mathbb{P}^\infty$.

We still need to prove this statement. In fact, we would like to show a generalization to k-dimensional bundles. For each k there is a real manifold, called the Grassmannian manifold and denoted by Gr_k, with a k-dimensional real vector bundle γ^k on Gr_k such that for any paracompact base space B the set of isomorphism classes of k-dimensional bundles over B is in bijection with the set of homotopy classes of maps $B \to \mathrm{Gr}_k$:

$$\mathrm{Vect}^k(B) \cong [B, \mathrm{Gr}_k].$$

The bundle γ^k is called the universal k-dimensional vector bundle.

11.2. **The Grassmannian.** The Grassmannian manifold $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the space of k-planes in $\mathbb{R}^{n+k}$. It can be identified with the quotient of the Stiefel manifold $V_k(\mathbb{R}^{n+k})$ of orthonormal sequences

$$[v_1, \ldots, v_k]$$

of vectors $v_i \in \mathbb{R}^{n+k}$, modulo the equivalence relation

$$[v_1, \ldots, v_k] \sim [v_1, \ldots, v_k] \cdot T,$$

with T any orthogonal $k \times k$-matrix.

Remark 11.1. The topology of the Stiefel manifold is given as follows. We can consider $V_k(\mathbb{R}^{n+k})$ as a subspace of the product $\mathbb{R}^{n+k} \times \ldots \times \mathbb{R}^{n+k}$ of k copies of $\mathbb{R}^{n+k}$. More precisely, $V_k(\mathbb{R}^{n+k})$ is the subspace of $S^{n+k-1} \times \ldots \times S^{n+k-1}$ of k copies of spheres S^{n+k-1} given by all orthogonal k-tuples. It is a closed subspace since orthogonality of two vectors can be expressed by an algebraic equation. In particular, $V_k(\mathbb{R}^{n+k})$ is compact, since the product of spheres is compact.

Now there is a natural surjective map

$$V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$$

sending an orthonormal sequence to the subspace it spans. We equip $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ with the quotient topology with respect to this surjection. In particular, $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is compact as well.

Example 11.2. We already know one example of a Grassmannian. The Grassmannian $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is just $\mathbb{P}^n$, and the presentation described above is just the representation of $\mathbb{P}^n$ as the quotient space of S^n by the antipodal action.

Proposition 11.3. *The space* $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ *is a manifold of dimension* $k \cdot n$*.*

Proof. Let $V \subset \mathbb{R}^{n+k}$ be a k-plane, and let W be the orthogonal complement of V. Then the subspace $U \subset \mathrm{Gr}_k(\mathbb{R}^{n+k})$ consisting of k-planes V' with the property that $V' \cap W = \{0\}$ is an open neighborhood of V.

Note: To see that U is open it suffices to show that its preimage $\tilde{U}$ in $V_k(\mathbb{R}^{n+k})$ is open. The set $\tilde{U}$ consists of all orthonormal frames $[v_1, \ldots, v_k]$ such that the $p(v_1), \ldots, p(v_k)$ are linearly independent where p is the projection

$$p\colon \mathbb{R}^{n+k} \to V.$$

Writing M for the $k \times k$-matrix with column vectors $p(v_1), \ldots, p(v_k)$ we see that $\tilde{U}$ consists of all frames such that the resulting M has non-zero determinant. Hence $\tilde{U}$ is an open subset.

Thinking of $V' \in U$ as the graph of a linear map $V \to W$, gives a bijection

$$T\colon U \to \mathrm{Hom}(V,W)$$

of U with $\mathrm{Hom}(V, W)$, which is a real vector space of dimension $k \cdot n$.

The correspondence T is in fact a homeomorphism. For let

$$p\colon V \oplus W \to V$$

be the orthogonal projection and let $x_1, \ldots, x_n$ be a fixed orthonormal basis for V. Then each V' in U has a unique basis $y_1, \ldots, y_n$ such that

$$p(y_1) = x_1, \ldots, p(y_n) = x_n.$$

The orthonormal frame $[y_1, \ldots, y_n]$ depends continuously on V'. Moreover, the $y_1, \ldots, y_n$ satisfy the identity

$$y_i = x_i + T(V')x_i \tag{1}$$

by definition of T and the choice of the y_i's. Hence, since y_i depends continuously on V', it follows that the image $T(V')x_i \in W$ depends continuously on V'. Therefore the linear transformation depends continuously on V'.

On the other hand the identity (1) shows that the n-frame $[y_1, \dots, y_n]$ depends continuously on $T(V')$, and hence that V' depends continuously on $T(V')$. Thus the function T^{-1} is also continuous and T is a homeomorphism. □

The inclusions $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+1+k} \subset \dots$ induce inclusions

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{R}^{n+1+k}) \subset \dots$$

The infinite Grassmannian manifold is the union

$$\mathrm{Gr}_k := \mathrm{Gr}(\mathbb{R}^\infty) = \bigcup_n \mathrm{Gr}_k(\mathbb{R}^{n+k}).$$

This is the set of all k-dimensional vector subspaces of $\mathbb{R}^\infty$. The topology of Gr_k is the direct limit topology, i.e., a subset of Gr_k is open (or closed) if and only if its intersection with $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is open (or closed) as a subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ for each n.

Once again, Gr_1 is just the infinite real projective space $\mathbb{P}^\infty$.

11.3. **The canonical bundle.** The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equipped with a canonical k-dimensional vector bundle $\gamma^k(\mathbb{R}^{n+k})$ defined as follows. Let

$$E = E(\gamma^k(\mathbb{R}^{n+k}))$$

be the set of all pairs

$$(k\text{-plane in } \mathbb{R}^{n+k},\ \text{vector in that } k\text{-plane}).$$

The topology on E is the topology as a subset of $\mathrm{Gr}_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$. The projection map

$$\pi\colon E \to \mathrm{Gr}_k(\mathbb{R}^{n+k}), \text{ is defined by } \pi(V, v) = V,$$

and the vector space structure is defined by

$$t_1(V, v_1) + t_2(V, v_2) = (V, t_1 v_1 + t_2 v_2).$$

Over the infinite Grassmannian Gr_k, there is also a canonical bundle γ^k whose total space is

$$E(\gamma^k) \subset \mathrm{Gr}_k \times \mathbb{R}^\infty$$

the set of all pairs

$$(k\text{-plane in } \mathbb{R}^\infty,\ \text{vector in that } k\text{-plane})$$

topologized as a subset of the product $\mathrm{Gr}_k \times \mathbb{R}^\infty$. The projection $\pi\colon E(\gamma^k) \to \mathrm{Gr}_k$ is given by $\pi(V, v) = V$.

Note that the bundles $\gamma^1(\mathbb{R}^{n+1})$ and γ^1 are of course just the bundles γ^1_n on $\mathbb{P}^n$ and γ^1 on $\mathbb{P}^n$ respectively.

Lemma 11.4. *The just constructed bundles* $\gamma^k(\mathbb{R}^{n+k})$ *and* γ^k *satisfy the local triviality condition.*

Proof. We start with $\gamma^k(\mathbb{R}^{n+k})$. Let $V \subset \mathbb{R}^{n+k}$ be a k-plane, and let U be the open neighborhood of V constructed in the proof of Proposition 11.3. The coordinate homeomorphism

$$h\colon U \times \mathbb{R}^k \cong U \times V \to \pi^{-1}(U)$$

is defined as follows. Set $h(V', x) := (V', y)$ where y denotes the unique vector in V' which is carried into x by the orthogonal projection

$$p\colon \mathbb{R}^{n+k} \to V.$$

The identities

$$h(V', x) = (V', x + T(V')x) \text{ and } h^{-1}(V', y) = (V', p(y))$$

show that h and h^{-1} are continuous.

For γ^k it suffices to note that an open neighborhood U for a k-plane V in Gr_k is just the union of the neighborhoods of V in the $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. Hence the coordinate homeomorphisms just constructed fit together to give a coordinate homeomorphism over U. The continuity follows from the fact that we use the direct limit topology on Gr_k. □

Our next goal is to prove the following fundamental result.

Theorem 11.5. *For a paracompact space* B*, the map* $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$*,* $[f] \mapsto f^*(\gamma^k)$*, is a bijection.*

Remark 11.6. The infinite Grassmannian Gr_k is called the *classifying space* and γ^k is called the *universal bundle* for k-dimensional real vector bundles.

12. Representability of $\mathrm{Vect}^k(B)$

Our next goal is to prove the following fundamental result.

Theorem 12.1. *For a paracompact space B, the map $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$, $[f] \mapsto f^*(\gamma^k)$, is a bijection.*

Remark 12.2. The theorem justifies to call the infinite Grassmannian Gr_k is the *classifying space* and γ^k is the *universal bundle* for k-dimensional real vector bundles.

Example 12.3. Let τ be the tangent bundle to S^n in $\mathbb{R}^{n+1}$. It is given by the projection $p\colon E(\tau) \to S^n$ where

$$E(\tau) = \{(x, v) \in S^n \times \mathbb{R}^{n+1} | x \perp v\}.$$

Each fiber $p^{-1}(x)$ is an n-plane and hence defines a point in $\mathrm{Gr}_n(\mathbb{R}^{n+1})$. This defines a map

$$S^n \to \mathrm{Gr}_n(\mathbb{R}^{n+1}),\ x \mapsto p^{-1}(x).$$

Via the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^\infty$ we can view this as a map

$$f\colon S^n \to \mathrm{Gr}_n(\mathbb{R}^\infty) = \mathrm{Gr}_n.$$

The bundle τ is exactly the pullback $f^*(\gamma^n)$. We check this on total spaces in the diagram

$$\begin{array}{ccc} E(\tau) \cong f^*(E(\gamma^n)) & \longrightarrow & E(\gamma^n) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S^n & \xrightarrow{\ f\ } & \mathrm{Gr}_n. \end{array}$$

since we have

$$f^*(E(\gamma^n)) = \{(x,(V,v)) \in S^n \times E(\gamma^n) | f(x) = \pi(V,v)\} = \{(x,(V,v)) | p^{-1}(x) = V, \text{i.e. } x \perp v\}.$$

12.1. **Proof of Theorem 16.5.** We first claim that, for a k-dimensional bundle $p\colon E = E(\xi) \to B$, an isomorphism $\xi \cong f^*(\gamma^k)$ is equivalent to a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber. To prove this claim suppose we have a map $f\colon B \to \mathrm{Gr}_k$ and an isomorphism $\xi \cong f^*(\gamma^k)$. Then we have a commutative diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\cong} & f^*(\gamma^k) & \xrightarrow{f'} & E(\gamma^k) & \xrightarrow{g^k} & \mathbb{R}^\infty \\ & {\scriptstyle p}\searrow & \downarrow & & \downarrow & & \\ & & B & \xrightarrow{f} & \mathrm{Gr}_k & & \end{array}$$

with $g^k(V,v) = v$. The composition along the top row is a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber, since both f' and g^k have this property. Conversely, given a map $g\colon E \to \mathbb{R}^\infty$ which is linear and injective on each fiber, define $f\colon B \to \mathrm{Gr}_k$ by letting $f(b)$ be the k-plane $g(p^{-1}(b))$. This yields a commutative diagram as above.

Now we are ready to prove the theorem. We start with the surjectivity of the map $[B, \mathrm{Gr}_k] \to \mathrm{Vect}^k(B)$. Let ξ be a k-dimensional bundle given by the map $p\colon E \to B$. Since B is paracompact there is a countable open cover $\{U_j\}$ of B such that ξ is trivial over each U_j and there is a partition of unity $\{\varphi_j\}$ with φ_j supported on U_j. Let $g_j\colon \pi^{-1}(U_j) \to \mathbb{R}^n$ be the composition of a trivialization $p^{-1}(U_j) \to U_j \times \mathbb{R}^n$ with the projection onto $\mathbb{R}^n$. The map

$$(\varphi_j \circ p) \cdot g_j\colon p^{-1}(U_j) \to \mathbb{R}^n,\ v \mapsto \varphi_j(p(v)) \cdot g_j(v)$$

extends to a map $E \to \mathbb{R}^n$ that is zero outside $p^{-1}(U_j)$. Near each point of B only finitely many φ_j's are nonzero, and at least one φ_j is nonzero. Hence these extended maps $(\varphi_j \circ p) \cdot g_j$ are the coordinates of a map $g\colon E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ that is a linear injection on each fiber. By our claim above this induces a map $f\colon B \to \mathrm{Gr}_k$ and the proof of surjectivity is complete.

For injectivity, let f_0, $f_1\colon B \to \mathrm{Gr}_k$ be two maps with isomorphisms $\xi \cong f_0^*(\gamma^k)$ and $\xi \cong f_1^*(\gamma^k)$. By our first claim these two maps induce maps g_0, $g_1\colon E \to \mathbb{R}^\infty$ which are linear and injective on each fiber. We will now show that g_0 and g_1 are homotopic through maps g_t which are linear and injective on each fiber. Then f_0 and f_1 are homotopic via

$$f_t(b) = g_t(p^{-1}(b)).$$

Therefore, let L_t be the homotopy

$$L_t\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ L_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(x_1, 0, x_2, 0, \ldots).$$

For each t, this is a linear map. Its kernel is trivial, since if

$$L_t(x_1, \ldots, x_n) = ((1-t)x_1 + tx_1, (1-t)x_2, (1-t)x_3 + tx_2, \ldots) = 0$$

then we get $x_1 = 0$, $x_2 = 0, \ldots$. Hence L_t is injective. Composing L_t with g_0 moves the image of g_0 into the odd-numbered coordinates and we have a homotopy which is linear and injective on fibers

$$g_0 = L_0 \circ g_0 \sim L_1 \circ g_0 =: \tilde{g}_0.$$

Similarly, let M_t be the homotopy

$$M_t\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,\ M_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(0, x_1, 0, x_2, 0, \ldots).$$

For each t, this is a linear map. Its kernel is trivial, since if

$$M_t(x_1, \ldots, x_n) = ((1-t)x_1, (1-t)x_2 + tx_1, (1-t)x_3, (1-t)x_4 + tx_2, \ldots) = 0$$

then we get $x_1 = 0$, $x_2 = 0, \ldots$. Hence M_t is injective. Composing M_t with g_1 moves the image of g_1 into the even-numbered coordinates and we have a homotopy which is linear and injective on fibers

$$g_1 = M_0 \circ g_1 \sim M_1 \circ g_1 =: \tilde{g}_1.$$

Then we let

$$\tilde{g}_t = (1-t)\tilde{g}_0 + t\tilde{g}_1.$$

The reason for composing with L_t and M_t is that $\tilde{g}_t$ is a map which is linear and injective on fibers for each t, since g_0 and g_1 are linear and injective on fibers. Overall we obtain homotopies which are linear and injective on fibers

$$g_0 \sim \tilde{g}_0 \sim \tilde{g}_1 \sim g_1$$

as desired. This completes the proof of Theorem 16.5.

12.2. **Universality reformulated.** The statement of Theorem 16.5 is closely related to the following two assertions which reformulate the universality of the canonical bundle γ^k.

Theorem 12.4. *For any k-dimensional bundle ξ over a paracompact base space B there exists a bundle map $f\colon \xi \to \gamma^k$.*

Proof. We have seen in the previous proof that there is a map

$$g\colon E(\xi) \to \mathbb{R}^\infty$$

which is linear and injective on the fibers of ξ and which is unique up to a homotopy which is linear and injective on the fibers. Then we can define the the bundle map f by

$$f(e) = (g(\text{fiber in which } e \text{ lies}), g(e)).$$

□

Two bundle maps $F, G\colon \xi \to \gamma^k$ are called *bundle-homotopic* if there exists a one-parameter family of maps

$$H_t\colon \xi \to \gamma^k,\ 0 \le t \le 1,$$

with $H_0 = F$, $H_1 = G$ such that

$$H\colon E(\xi) \times [0,1] \to E(\gamma^k)$$

is continuous as a function of both variables.

Theorem 12.5. *Any two bundle maps from a k-dimensional bundle ξ to γ^k are bundle-homotopic.*

Proof. Let ξ be given by the map $p\colon E \to B$. We know that a bundle map $F\colon \xi \to \gamma^k$ determines a map

$$g\colon E(\xi) \to \mathbb{R}^\infty$$

whose restriction to each fiber of ξ is linear and injective. Conversely, g determines F by the identity

$$F(e) = (g(\text{fiber in which } e \text{ lies}), g(e)).$$

Now suppose we have two bundle maps $F_0, F_1\colon \xi \to \gamma^k$ and let $f_0, f_1\colon B \to \mathrm{Gr}_k$ be the corresponding maps on base spaces. We have seen in Lecture 04 that the bundle maps F_0, F_1 come equipped with isomorphisms $\xi \cong f_0^*(\gamma^k)$ and $\xi \cong f_1^*(\gamma^k)$. Then we know from the proof of Theorem 16.5 that there is a homotopy g_t between g_0 and g_1 which induces a homotopy f_t between f_0 and f_1. But the homotopy g_t also induces a bundle homotopy F_t between F_0 and F_1 by defining

$$F_t(e) := (g_t(\text{fiber in which } e \text{ lies}), g_t(e)).$$

□

12.3. **Universal characteristic classes.** We can use the above results to reconsider the concept of characteristic classes. For a k-dimensional vector bundle ξ let $f_\xi\colon B \to \mathrm{Gr}_k$ be a representative of the homotopy class corresponding to ξ under the bijection of Theorem 16.5.

Now let R be any coefficient ring and let

$$c \in H^i(\mathrm{Gr}_k; R)$$

be any cohomology class. Then we get an induced class

$$c(\xi) := f_\xi^*(c) \in H^i(B; R).$$

Definition 12.6. The class $c(\xi)$ is called the *characteristic cohomology class* of ξ determined by c.

Note that the correspondence $\xi \mapsto c(\xi)$ is natural with respect to bundle maps, i.e., it commutes with pullbacks.

Conversely, given any correspondence

$$\xi \mapsto c(\xi) \in H^i(B; R)$$

which is natural with respect to bundle maps, then we must have

$$c(\xi) = f_\xi^* c(\gamma^k).$$

Thus the above construction is the most general one. In other words:

Corollary 12.7. *The ring consisting of all characteristic cohomology classes for k-dimensional bundles over paracompact base spaces with coefficient ring R is canonically isomorphic to the cohomology ring $H^*(\mathrm{Gr}_k; R)$.*

Hence it is a very important task to compute the cohomology ring $H^*(\mathrm{Gr}_k; R)$. For $R = \mathbb{Z}/2$, we will do this in the next lecture.

13. Schubert cells and Schubert varieties

In this lecture we follow notes by Mike Hopkins which are not listed in the references mentioned at the beginning of the semester.

The interior of the i-cell in $\mathbb{P}^n$ is the space of lines contained in $\mathbb{R}^{i+1}$ but not in $\mathbb{R}^i$. There is an analogous cell decomposition of the Grassmannian. Each k-plane $V \subset \mathbb{R}^{n+k}$ determines a sequence of numbers

$$(\dim V \cap \mathbb{R}^1, \dim V \cap \mathbb{R}^2, \ldots).$$

Note that the dimension jumps in each step by at most one, since the following sequence is exact:

$$0 \to V \cap \mathbb{R}^{i-1} \to V \cap \mathbb{R}^i \xrightarrow{i\text{-th coordinate}} \mathbb{R}.$$

Moreover, he sequence contains exactly k jumps.

For instance, if V is the 3-plane in $\mathbb{R}^5$ spanned by the rows of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 \end{pmatrix}$$

then our sequence of numbers would be

$$(0, 1, 2, 2, 3).$$

Let us keep track of where the dimensions jump, and record these numbers as $(j_1, \ldots, j_k)$. In our example the sequence of j's would be

$$(2, 3, 5).$$

Finally, for reasons that will be clear in a moment, we decide to use the sequence $(a_1, \ldots, a_k)$ instead with

$$a_i = j_i - i.$$

In our example the sequence of a's is

$$(1, 1, 2).$$

Definition 13.1. A *Schubert symbol* is a sequence $\underline{a} = (a_1, \ldots, a_k)$, with

$$0 \leq a_1 \leq \ldots \leq a_k.$$

The associated *jump sequence* is the sequence $\underline{j} = (j_1, \ldots, j_k)$ with $j_i = a_i + i$.

Remark 13.2. One should be aware of that other authors also use the name "Schubert symbol" to refer to the sequence $\underline{j}$.

Now let $\underline{a} = (a_1, \ldots, a_k)$ with $a_k \leq n$ be a Schubert symbol, and let

$$H_i := \mathbb{R}^{j_i}$$

with $j_i = a_i + i$ as before. Then the H_i define a filtration of $\mathbb{R}^{n+k}$

$$0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subseteq \mathbb{R}^{n+k}.$$

We set

$$\Omega_{\underline{a}} = \{V \in \mathrm{Gr}_k(\mathbb{R}^{n+k}) | \dim V \cap H_i \geq i\}.$$

Definition 13.3. The space $\Omega_{\underline{a}}$ is called the *Schubert variety* associated to the Schubert symbol $\underline{a}$.

Example 13.4. When $k = 1$ the sequence $\underline{a}$ is just a number a. In that case the Schubert variety is $\mathbb{P}^a$.

As a next step we will make the set of Schubert symbols of a fixed length k into a partially ordered set by defining $\underline{a}' \leq \underline{a}$ if and only if

$$a'_i \leq a_i \text{ for } i = 1, \ldots, k.$$

We can use this ordering to make the set of *all* Schubert symbols into a partially ordered set by first filling the symbols on the left with 0's to make them have the same length, and then using the above partial ordering. Thus, with this convention

$$(1, 2, 2) \geq (1, 2),$$

since

$$(1, 2, 3) \geq (0, 1, 2).$$

Definition 13.5. The *Schubert cell* associated to the Schubert symbol $\underline{a}$ is the space

$$\Omega^0_{\underline{a}} = \Omega_{\underline{a}} - \bigcup_{\underline{a}' < \underline{a}} \Omega_{\underline{a}'}.$$

Remark 13.6. Another warning: The Schubert cells are not quite "cells". They are merely the interiors of cells.

Remark 13.7. The space $\Omega^0_{\underline{a}}$ consists exatly of the $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ whose associated Schubert symbol is $\underline{a}$. In particular, each V lies in exactly one $\Omega^0_{\underline{a}}$ where $\underline{a}$ is the Schubert symbol corresponding to the dimension sequence of V.

Proposition 13.8. *The space $\Omega^0_{\underline{a}}$ is homeomorphic to $\mathbb{R}^{|\underline{a}|}$, where we denote $|\underline{a}| = a_1 + \ldots + a_k$.*

Proof. We show that each $V \in \Omega^0_{\underline{a}}$ has a canonical basis of a special form. Let $\{\epsilon_1, \ldots, \epsilon_{n+k}\}$ be the standard basis of $\mathbb{R}^{n+k}$. First, choose a non-zero $v_1 \in V \cap H_1$. This space is one-dimensional, so v_1 is determined up to a scalar mutliple. We can normalize v_1 by requiring $\langle \epsilon_{j_1}, v_1\rangle = 1$. Now choose $v_2 \in V \cap H_2$ with the properties

$$\begin{aligned}\langle \epsilon_{j_2}, v_2\rangle &= 1\\ \langle \epsilon_{j_1}, v_2\rangle &= 0.\end{aligned}$$

Since $V \cap H_2$ has dimension 2 these two equations characterize v_2 uniquely, provided they can be solved. But we know they can be solved. For the map

$$V \cap H_2 \to H_2 \to H_2/\mathbb{R}^{j_2-1}\mathbb{R}\cdot\epsilon_{j_2}$$

is non-zero since $\dim V \cap \mathbb{R}^{j_2-1} = 1$. Continuing, we find a unique basis $\{v_1, \ldots, v_k\}$ of V with the property that $v_i \in H_i$ for all i, and

$$\begin{aligned}\langle \epsilon_{j_s}, v_s\rangle &= 1 \text{ for all } s \text{ and}\\ \langle \epsilon_{j_s}, v_t\rangle &= 0 \text{ for } s \neq t.\end{aligned}$$

Now if we let V vary in $\Omega^0_{\underline{a}}$, we see that the space of all possible v_i's is a vector space of dimension $\dim H_i - i$, since v_i lies in H_i and has to satisfy i equations. □

Remark 13.9. Another way to think of the v_i is to consider them as the rows in a matrix. For example, in the case $\mathrm{Gr}_3(\mathbb{R}^{4+3})$, with $\underline{a} = (2, 3, 4)$ such a matrix takes the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0\\ * & * & 0 & * & 1 & 0 & 0\\ * & * & 0 & * & 0 & * & 1\end{pmatrix}$$

where the $*$'s denote arbitrary numbers as entries. The rows of this matrix are the vectors v_1, v_2, and v_3. Hence we see that the decomposition of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ into Schubert cells corresponds to taking a matrix, reducing it to row echelon form, and recording the columns with the pivots.

14. A cell decomposition for the Grassmannian

Recall from the previous lecture:

- *Schubert symbols*: sequences $\underline{a} = (a_1, \ldots, a_k)$, with $0 \leq a_1 \leq \ldots \leq a_k$. The associated *jump sequence* is the sequence $\underline{j} = (j_1, \ldots, j_k)$ with $j_i = a_i + i$.
- For given $\underline{a}$, filtration $0 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subseteq \mathbb{R}^{n+k}$ with $H_i := \mathbb{R}^{j_i}$.
- The *Schubert variety* $\Omega_{\underline{a}} = \{V \in \mathrm{Gr}_k(\mathbb{R}^{n+k}) | \dim V \cap H_i \geq i\}$ associated to $\underline{a}$.
- The *Schubert cell* $\Omega^0_{\underline{a}} = \Omega_{\underline{a}} - \bigcup_{\underline{a}' \leq \underline{a}} \Omega_{\underline{a}'}$ associated to $\underline{a}$.
- We proved that the space $\Omega^0_{\underline{a}}$ is homeomorphic to $\mathbb{R}^{|\underline{a}|}$, where we denote $|\underline{a}| = a_1 + \ldots + a_k$. We did this by showing that each $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ has a special basis and the space of choices of those bases is a vector space of dimension $|\underline{a}|$.

We will use these notions and the above result to define a CW-decomposition of the Grassmannian manifold. We still follow the notes by Mike Hopkins.

14.1. **A CW-decomposition.** To see that the Schubert cells serve as the cells of a CW-decomposition, we need to define the characteristic maps. For each $\underline{a}$ let $D^{\underline{a}} \subset V_k(\mathbb{R}^{n+k})$ be the set of orthonormal sequences $(v_1, \ldots, v_k)$ satisfying

$$\begin{aligned} v_i &\in H_i \\ \langle \epsilon_i, v_i \rangle &\geq 0. \end{aligned}$$

We define a map

$$s_{\underline{a}} \colon D^{\underline{a}} \to \Omega_{\underline{a}}$$

by sending $(v_1, \ldots, v_k)$ to the plane it spans.

Lemma 14.1. *The map $s_{\underline{a}}$ restricts to a homeomorphism of the interior of $D^{\underline{a}}$ with $\Omega^0_{\underline{a}}$.*

Proof. Let $s^0_{\underline{a}}$ be the restriction of $s_{\underline{a}}$ to the interior of $D^{\underline{a}}$. Let $(v_1, \ldots, v_k)$ be an orthonormal frame on the boudnary of $D^{\underline{a}}$. Then

$$V := s^0_{\underline{a}}((v_1, \ldots, v_k))$$

does not belong to $\Omega^0_{\underline{a}}$, for one of the vectors v_i must have $j_i - 1$th component equal to 0. This implies

$$\dim(V \cap \mathbb{R}^{j_i - 1}) \geq i,$$

since we have $\dim(V \cap \mathbb{R}^{j_i}) \geq i$. Hence V does not lie in $\Omega^0_{\underline{a}}$, since for a k-plane in $\Omega^0_{\underline{a}}$ the number j_i is exactly the first dimension where $V \cap \mathbb{R}^m$ has dimension i. The construction of the previous lecture of the special basis for the planes in

$\Omega^0_{\underline{a}}$ then shows that $s^0_{\underline{a}}$ is a bijection. It remains to show that $s^0_{\underline{a}}$ and its inverse are continuous. We leave this to the reader. □

The next result shows that the $s_{\underline{a}}$ serve as characteristic maps for the cells in the Grassmannian.

Proposition 14.2. *The space $D^{\underline{a}}$ is homeomorphic to the product*

$$D_0^{a_1} \times D_0^{a_2} \times \ldots \times D_0^{a_k},$$

in which each $D_0^{a_i}$ is the disk consisting of the unit vectors $v \in H_i$ with the properties

$$\begin{aligned} \langle v, \epsilon_{j_i} \rangle &\geq 0 \\ \langle v, \epsilon_{j_t} \rangle &= 0 \text{ for } t < i. \end{aligned}$$

Hence $D^{\underline{a}}$ is homeomorphic to the disk $D^{a_1+\cdots+a_k}$.

Proof. For each unit vector $v \in H_1$ with $\langle \epsilon_{j_i}, v\rangle \geq 0$, let $T_v \in SO(n+k)$ be the orthogonal transformation which rotates v to ϵ_{j_1} in the plane spanned by v and ϵ_{j_1}, and which is the identity on the orthogonal complement of this plane. Note that T_v restricts to an orthogonal transformation of H_i to itself since both v and ϵ_{j_i} are in H_i (H_1 is a subspace of H_i), and has the property that $T_v(\epsilon_{j_i}) = \epsilon_{j_i}$ for $i > 1$, since both v and ϵ_{j_1} are orthogonal to ϵ_{j_i}. We now use this transformation T to define a homeomorphism

$$D^{\underline{a}} \to D_0^{a_1} \times D_1^{\underline{a}'}, \tag{2}$$

in which $D_1^{\underline{a}'}$ is the space of orthonormal sequences

$$(v_2', \ldots, v_k')$$

with $v_i' \in H_i \cap \{\epsilon_{j_1}\}^\perp$, and

$$\langle \epsilon_i, v_i' \rangle \geq 0.$$

In other words, $D_1^{\underline{a}'}$ is the cell in $\mathrm{Gr}_{k-1}(\mathbb{R}^{n+k-1})$ associated to the sequence

$$\underline{a}' = (a_2, \ldots, a_k),$$

in which we are regarding $\mathbb{R}^{n+k-1}$ as the Euclidean space with basis

$$\{\epsilon_t | t \neq j_1\}.$$

Once we establish the homeomorphism (2), we are done by induction on k.

The homeomorphism (2) is the map whose first component is the projection

$$(v_1, \ldots, v_k) \mapsto v_1,$$

and whose second component is

$$(T_{v_1} v_2, \ldots, T_{v_1} v_k),$$

so that

$$v'_i = T_{v_1} v_i.$$

Since T_{v_1} is orthogonal, the sequence $(v'_2, \dots, v'_k)$ is orthonormal. To verify the conditions that the sequence be in $D_1^{\underline{a}'}$, first note that for $i > 1$, we have

$$0 = \langle v_1, v_i \rangle = \langle T_{v_1} v_1, T_{v_1} v_i \rangle = \langle \epsilon_{j_1}, T_{v_1} v_i \rangle,$$

and also

$$0 \le \langle \epsilon_{j_1}, v_i \rangle = \langle T_{v_1} \epsilon_{j_1}, T_{v_1} v_i \rangle = \langle \epsilon_{j_1}, T_{v_1} v_i \rangle,$$

since ϵ_i is orthogonal to both ϵ_{j_1} and v_1. The inverse homeomorphism is

$$(v_1, v'_2, \dots, v'_k) \mapsto (v_1, T_{v_1}^{-1} v'_2, \dots, T_{v_1}^{-1} v'_k).$$

Reversing the above computations which checked the conditions shows that it carries $D_0^{a_1} \times D_1^{\underline{a}'}$ to $D^{\underline{a}}$. $\square$

Remark 14.3. a) There are $\binom{n+k}{k}$ cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. This is the number of ways of choosing k distinct numbers j_i with $j_i \le n+k$.
b) In particular, the number of r-cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equal to the number of partitions of r into at most k integers a_i each of which is $\le n$.
c) If k and n are $\ge r$ then the number of r-cells in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is equal to the number of partitions of r into at most k integers (zeroes in the beginning of the sequence $\underline{a}$ are allowed).
d) The number of r-cells in Gr_k is equal to the number of partitions of r into at most k integers.

Corollary 14.4. *The maps*

$$s_{\underline{a}'} \colon D^{\underline{a}'} \to \Omega_{\underline{a}}$$

with $\underline{a}' \le \underline{a}$ are the characteristic maps of the cells in a CW-decomposition of the Schubert variety $\Omega_{\underline{a}}$.

In the next lecture we will prove the following result.

Proposition 14.5. *The cellular boundary map*

$$d^{cell} \colon C_*^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2 \to C_{*-1}^{cell}(\Omega_{\underline{a}}) \otimes \mathbb{Z}/2$$

is zero.

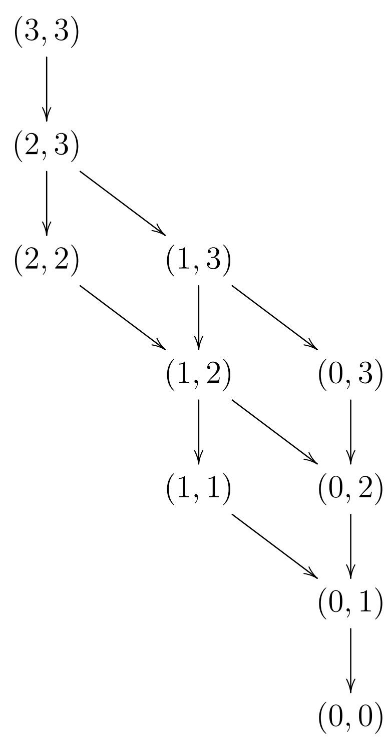
Let $x_{\underline{a}}$ be the homology class corresponding to the cellular cycle given by the map $s_{\underline{a}}$. Then the above result implies the following fundamental fact.

Corollary 14.6. *The classes*

$$x_{\underline{a}'} \in H_{\underline{a}'}(\Omega_{\underline{a}}; \mathbb{Z}/2)$$

with $\underline{a}' \le \underline{a}$ form a basis for the homology groups, where $|\underline{a}| = a_1 + \ldots + a_k$.

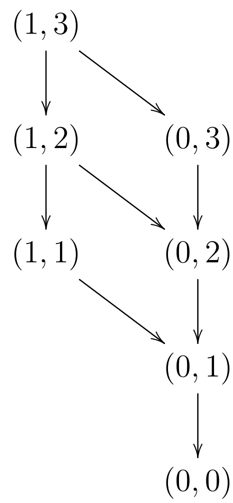
Before we prove these results, we look at some consequences. The picture below lists the sequences $\underline{a}$ occurring in the cell decomposition of $\mathrm{Gr}_2(\mathbb{R}^{3+2})$. The reverse of the partial ordering is indicated by an arrow, and the height corresponds to the dimension of the cell: (Recall: The dimension of $\mathrm{Gr}_2(\mathbb{R}^{3+2})$ is 6, the Schubert symbol $(3,3)$ has associated the maximal jump sequence (4,5) and corresponds to a cell in dimension $3+3=6$. The cell $(0,0)$ is in dimension zero.)



By looking at this diagram we see that the homology satisfies Poincaré duality in the sense that

$$\dim H_i(\mathrm{Gr}_2(\mathbb{R}^5)) = \dim H_{6-i}(\mathrm{Gr}_2(\mathbb{R}^5)).$$

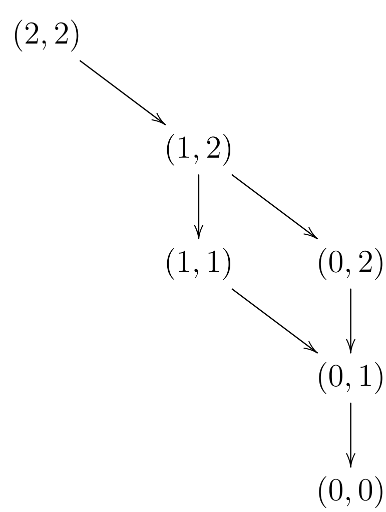
For instance, if we want the homology of $\Omega_{(1,3)}$ we look at the position labeled $(1,3)$, and everything below it



We can see from the diagram that $\Omega_{(1,3)}$ cannot satisfy Poincaré duality,

$$\dim H_i(\Omega_{\underline{a}}) = \dim H_{|\underline{a}|-i}(\Omega_{\underline{a}}). \tag{3}$$

Hence $\Omega_{(1,3)}$ cannot be a manifold. Looking at the diagram, the only Schubert varieties in $\mathrm{Gr}_2(\mathbb{R}^5)$ which might be manifold are $\Omega_{(2,2)}$ with



and $\Omega_{(0,i)}$ with $i \leq 3$ and

$$\begin{array}{c} (0,3) \\ \downarrow \\ (0,2) \\ \downarrow \\ (0,1) \\ \downarrow \\ (0,0) \end{array}$$

In fact, one can show that if the homology of $\Omega_{\underline{a}}$ satisfies Poincaré duality in the sense of (3) then $\Omega_{\underline{a}}$ is homeomorphic to $\mathrm{Gr}_\ell(\mathbb{R}^{m+\ell})$ for some pair (ℓ, m) and so is in fact a manifold. The point is that the Poincaré duality condition implies that the Schubert symbol $\underline{a}$ must have exactly one immediate predecessor. (You will be asked to prove this on the next Problem Set.)

15. The cohomology of the Grassmannian

Our first goal is to show the following result.

Proposition 15.1. *The cellular boundary map*

$$d^{cell}\colon C_*^{cell}(\Omega_{\underline{a}})\otimes\mathbb{Z}/2\to C_{*-1}^{cell}(\Omega_{\underline{a}})\otimes\mathbb{Z}/2$$

is zero.

Let $x_{\underline{a}}$ be the homology class corresponding to the cellular cycle given by the map $s_{\underline{a}}\colon D^{\underline{a}}\to\Omega_{\underline{a}}$ defined in the previous lecture. Then the above result implies the following fundamental fact.

Corollary 15.2. *The classes*

$$x_{\underline{a}'}\in H_{\underline{a}'}(\Omega_{\underline{a}};\mathbb{Z}/2)$$

with $\underline{a}'\le\underline{a}$ form a basis for the homology groups, where $|\underline{a}|=a_1+\ldots+a_k$.

15.1. **The flag varieties.** The aim of this section is to prove Proposition 15.1. Therefore, we start with an observation. Suppose that X is a CW-complex, M is a closed manifold of dimension n, and $f\colon M\to X^{(n)}$ is a map form M to the n-skeleton of X. Let $\alpha_M\in H_n(M;\mathbb{Z}/2)$ be the fundamental class. The image of α_M under the map

$$H_n(M)\to H_n(X^{(n)})\to H_n(X^{(n)},X^{(n-1)})=C_n^{cell}(X)$$

defines a cellular chain $c_M\in C_n^{cell}(X)$. In fact this chain is a cycle since it lies in the image of $H_n(X^{(n)})$ and so goes to zero under the first map in the factorization

$$H_n(X^{(n)},X^{(n-1)})\to H_{n-1}(X^{n-1})\to H_{n-1}(X^{(n-1)},X^{(n-2)})$$

of the cellular boundary map. In this way, maps of manifolds give homology classes, and, in fact cycles in the complex of cellular chains.

We will need to be able to specify the cycle we constructed more precisely. If the map

$$f\colon M\to X':=X^{(n-1)}\cup D_\alpha^n\subset X^{(n)},$$

and that for some point x in the interior of D_α^n there is a neighborhood U of x, contained in the interior of D_α^n, with the property that the restriction of f is a homeomorphism

$$f^{-1}(U)\to U.$$

In that case, the diagram

$$\begin{array}{ccccc} H_n(M) & \xrightarrow{\approx} & H_n(M, M - f^{-1}(x)) & & \\ \downarrow & & \downarrow \approx & & \\ H_n(X') & \longrightarrow & H_n(X', X' - \{x\}) & & \\ & & \downarrow \approx & & \\ & & H_n(D^n_\alpha, S^{n-1}_\alpha) & \longrightarrow & C^{cell}_n(X) \end{array}$$

shows that the cellular cycle c_M is just the chain represented by the cell D^n_α. In particular, one learns in this case that the cellular represented by D^n_α is, in fact, a cycle. We will use these ideas to prove Proposition 15.1.

For each $\underline{a}$, let

$$F_{\underline{a}} \subset \mathrm{Gr}_1(H_1) \times \cdots \times \mathrm{Gr}_k(H_k)$$

be the subspace consisting of sequences $(V_1, \dots, V_k)$ with

$$V_1 \subset V_2 \subset \cdots \subset V_k.$$

For some purposes it is useful to note that $F_{\underline{a}}$ can also be identified with the space

$$F_{\underline{a}} \subset \mathbb{P}(H_1) \times \cdots \times \mathbb{P}(H_k)$$

consisting of sequences of lines $(\ell_1, \dots, \ell_k)$ which are pairwise orthogonal. There is an obvious homeomorphism between these, under which V_j corresponds to $\ell_1 \oplus \cdots \oplus \ell_j$, and ℓ_j to the orthogonal complement of V_{j-1} in V_j.

Proposition 15.3. *The space $F_{\underline{a}}$ is a manifold.*

Proof. The proof is very similar to the proof of Proposition 11.3. Let

$$\begin{array}{ccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_k \\ \downarrow & & \downarrow & & & & \downarrow \\ H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots & \longrightarrow & H_k \end{array} \tag{4}$$

be a point in $F_{\underline{a}}$, and write W_i for the orthogonal complement of V_i in H_i. By identifying W_i with the quotient space H_i/V_i, the W_i fit into a sequence

$$W_1 \to W_2 \to \cdots \to W_k.$$

(This sequence is not, in general, a sequence of monomorphisms.)

Let $U \subset F_{\underline{a}}$ be the open neighborhood of the point (4) consisting of sequences $(V_1' \subset \cdots \subset V_k')$ with the property that for all i, $V_i' \cap W_i = \{0\}$. For such a sequence, we may think of V_i' as the graph of a homomorphism $V_i \to W_i$. This correspondence gives a homeomorphism of U with the space of sequences of linear maps $V_i \to W_i$ fitting into a diagram

$$\begin{array}{ccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_k \\ \downarrow & & \downarrow & & & & \downarrow \\ W_1 & \longrightarrow & W_2 & \longrightarrow & \cdots & \longrightarrow & W_k \end{array}$$

By choosing a basis $\{v_1, \ldots, v_k\}$ of V_k with $v_i \in V_i$ one can identify this space with

$$W_1 \oplus \cdots \oplus W_k.$$

Hence this is a vector space with of dimension

$$\dim W_1 + \cdots + \dim W_k = a_1 + \cdots + a_k.$$

□

Now let

$$f_{\underline{a}} \colon F_{\underline{a}} \to \Omega_{\underline{a}}$$

be the map sending a sequence $(V_1, \ldots, V_k)$ to V_k.

Proposition 15.4. *The map*

$$f_{\underline{a}}^{-1}(\Omega_{\underline{a}}^0) \to \Omega_{\underline{a}}^0$$

is a homeomorphism.

Proof. The inverse map sends $V \in \Omega_{\underline{a}}^0$ to the sequence $(V_1, \ldots, V_k)$ in which $V_i = V \cap H_i$. □

Now are finally ready to prove Proposition 15.1. The Schubert cell of $\Omega_{\underline{a}}$ has one cell of dimension $a_1 + \cdots + a_k$ and all other cells of lower dimension. We just proved that $F_{\underline{a}}$ is a manifold. Hence the argument described at the beginning of this section applied to the map

$$F_{\underline{a}} \to \Omega_{\underline{a}},$$

shows that the corresponding chain is a cycle. This shows that the boundary map d^{cell} vanishes on the one cell in dimension $|\underline{a}|$. All other elements in the cell complex are given by maps from cells $D^{\underline{a}'}$ for $\underline{a}' < \underline{a}$ to $\Omega_{\underline{a}}$. It follows from the ordering of the Schubert cells and the definition of Schubert varieties that the map $s_{\underline{a}'} \colon D^{\underline{a}'} \to \Omega_{\underline{a}}$ factors through the map $\Omega_{\underline{a}'} \to \Omega_{\underline{a}}$. This shows that the boundary map d^{cell} actually vanishes on all elements in $C_*^{cell}(\Omega) \otimes \mathbb{Z}/2$. This completes the proof of Proposition 15.1.

15.2. **The cohomology ring $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$.** We will finally determine the cohomology ring of the Grassmannian manifold Gr_k.

Theorem 15.5. *The cohomology ring $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$ is a polynomial algebra over $\mathbb{Z}/2$ freely generated by the Stiefel-Whitney classes $w_1(\gamma^k),\dots,w_k(\gamma^k)$.*

The idea of the proof is to show first that the Stiefel-Whitney classes of the canonical bundle over Gr_k freely generate a polynomial algebra over $\mathbb{Z}/2$ contained in $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$. Our knowledge about the cell structure of Gr_k then allows us to show that $H^*(\mathrm{Gr}_k;\mathbb{Z}/2)$ is actually equal to this polynomial algebra.

We start with the following lemma.

Lemma 15.6. *There are no polynomial relations among the $w_i(\gamma^k)$.*

Proof. Suppose that there is a relation of the form $p(w_1(\gamma^k),\dots,w_k(\gamma^k)) = 0$, where p is a polynomial in k variables over $\mathbb{Z}/2$. By the naturality of Stiefel-Whitney classes, for any k-dimensional bundle ξ over a paracompact base space there exists a bundle map $g\colon \xi \to \gamma^k$. If we denote the induced map on base spaces by $\bar{g}$ we get

$$w_i(\xi) = \bar{g}^*(w_i(\gamma^k)).$$

It follows that the cohomology classes $w_i(\xi)$ must satisfy the corresponding relation

$$p(w_1(\xi),\dots,w_k(\xi)) = \bar{g}^* p(w_1(\gamma^k),\dots,w_k(\gamma^k)) = 0.$$

Thus to prove the lemma it suffices to find some k-dimensional bundle ξ such that there are no polynomial relations among the classes $w_1(\xi),\dots,w_k(\xi)$.

Let γ^1 be the canonical line bundle over $\mathbb{P}^\infty = \mathrm{Gr}_1$. We know that $H^*(\mathbb{P}^\infty;\mathbb{Z}/2)$ is a polynomial algebra over $\mathbb{Z}/2$ with one generator a of dimension one and $w(\gamma^1) = 1 + a$. Taking the k-fold product

$$X := \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty,$$

it follows that $H^*(X;\mathbb{Z}/2)$ is a polynomial algebra on k generators $a_1,\dots,a_k$ of dimension one. Here a_i can be defined as the image $\pi_i^*(a)$ induced by the projection map

$$\pi_i\colon X \to \mathbb{P}^\infty$$

to the ith factor. We define ξ to be the k-fold product

$$\xi = \gamma^1 \times \cdots \times \gamma^1 \cong (\pi_1^*\gamma^1) \oplus \cdots \oplus (\pi_k^*\gamma^1).$$

Then ξ is a k-dimensional bundle over X, and the total Stiefel-Whitney class

$$w(\xi) = \pi_1^*(w(\gamma^1)) \cdot \cdots \cdot \pi_k^*(w(\gamma^1)) = (1 + a_1)(1 + a_2)\cdots(1 + a_k).$$

Hence $w_i(\xi)$ is the ith elementary symmetric function of $a_1,\dots,a_k$. It is a well-known theorem in algebra that the k elementary symmetric functions in

k variables over a field do not satisfy any polynomial relations. Thus the classes $w_1(\xi), \ldots, w_k(\xi)$ are algebraically independent over $\mathbb{Z}/2$, and it follows that the $w_1(\gamma^k), \ldots, w_k(\gamma^k)$. □

Now let us turn to the proof of Theorem 16.6. By the previous lemma, we know that $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$ contains a polynomial algebra over $\mathbb{Z}/2$ freely generated by $w_1(\gamma^k), \ldots, w_k(\gamma^k)$. We will show that $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$ actually coincides with this sub-algebra.

We know from the discussion of the cell discussion of Gr_k is equal to the number of partitions of r into at most k integers. Hence the dimension of $H^r(\mathrm{Gr}_k; \mathbb{Z}/2)$ is at most equal to this number of partitions. On the other hand, we claim that the number of distinct monomials of the form

$$w_1(\gamma^k)^{r_1} \cdots w_k(\gamma^k)^{r_k}$$

in $H^r(\mathrm{Gr}_k; \mathbb{Z}/2)$ is also precisely equal to the number of partitions of r into at most k integers. For to each sequence $r_1, \ldots, r_k$ of non-negative integers with

$$r_1 + 2r_2 + \cdots + kr_k = r \tag{5}$$

we can associate the partition of r which is obtained from the k-tuple

$$r_k, r_k + r_{k-1}, \ldots, r_k + r_{k-1} + \cdots + r_1 \tag{6}$$

by deleting any zeros which may occur. Conversely, to a partition (6) corresponds a sequence $r_1, \ldots, r_k$ of non-negative integers satisfying (5).

Since $\mathbb{Z}/2[w_1(\gamma^k), \ldots, w_k(\gamma^k)]$ is a sub-algebra of $H^*(\mathrm{Gr}_k; \mathbb{Z}/2)$, comparing the degrees and dimensions proves the theorem.

16. Chern classes for complex vector bundles

16.1. **Orientation.** From now on we will shift our focus to complex vector bundles. Much of the theory for real vector bundles carries over to the complex case. But there are a couple of important features of complex bundles. The first one is that the complex structure induces an orientation of the underlying real bundle.

Lemma 16.1. *Let ω be a complex vector bundle. Then the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonical preferred orientation.*

Proof. Let V be a finite dimensional complex vector space. Choosing a basis $a_1, \ldots, a_n$ for V over $\mathbb{C}$, gives us a real basis for the underlying real vector space $V_{\mathbb{R}}$:

$$a_1, ia_1, a_2, ia_2, \ldots, a_n, ia_n.$$

We claim that this ordered basis determines the required orientation for $V_{\mathbb{R}}$. For if $b_1, \ldots, b_n$ is any other complex basis of V, then there is a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ which transforms the first basis into the second. This deformation does not alter the orientation of the real vector space, since if $A \in \mathrm{GL}_n(\mathbb{C})$ is the coordinate change matrix, then the underlying real matrix $A_{\mathbb{R}} \in \mathrm{GL}_{2n}(\mathbb{R})$ has determinant

$$\det A_{\mathbb{R}} = |\det A|^2 > 0.$$

Hence $A_{\mathbb{R}}$ preserves the orientation of the underlying real vector space. Another way to see this is to note that $GL_n(\mathbb{C})$ is connected. Hence we can pass from any given complex basis to any other basis by a continuous deformation, and this continuous deformation cannot alter the orientation.

Now if ω is a complex vector bundle, then applying this construction to every fiber of ω yields the required orientation for $\omega_{\mathbb{R}}$, since overlapping trivializations determine a section in $\mathrm{GL}_n(\mathbb{C})$. □

Remark 16.2. As a consequence, every complex manifold is oriented, since an orientation of the tangent bundle of a manifold induces an orientation of the manifold itself.

16.2. **Chern classes.** Chern classes for complex vector bundles can be characterized by almost the same set of axioms as Stiefel–Whitney classes.

Theorem 16.3. *There is a unique sequence of functions $c_1, c_2, \ldots$ assigning to each complex vector bundle $E \to B$ over a a space B a class $c_i(E) \in H^{2i}(B;\mathbb{Z})$, depending only on the isomorphism type of E, such that*

a) $c_i(f^*E) = f^*(c_i(E))$ for a pullback along a map $f \colon B' \to B$ which is covered by a bundle map.
b) $c(E_1 \oplus E_2) = c(E_1)c(E_2)$ where $c = 1 + c_1 + c_2 + \ldots \in H^*(B;\mathbb{Z}/2)$.

c) $c_i(E) = 0$ *if* $i > \dim E$.

d) For the canonical complex line bundle γ_1^1 *on* $\mathbb{C}\mathrm{P}^\infty$, $c_1(\gamma_\infty^1)$ *is a specified generator of* $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

Proof. The proof is almost the same as for the existence and uniqueness of Stiefel–Whitney classes with $\mathbb{Z}$-coefficients and $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$. The bundle E induces a map $g\colon E \to \mathbb{C}^\infty$ which is linear and injective on fibers. Define $x \in H^2(E;\mathbb{Z})$ to be the element $\mathbb{C}\mathrm{P}(g)^*(\alpha)$. The Leray–Hirsch theorem applied to the fiber bundle $\mathbb{C}\mathrm{P}(E) \to B$ then implies that the elements $1, x, \ldots, x^{n-1}$ form a basis of $H^*(\mathbb{C}\mathrm{P}(E);\mathbb{Z})$ as an $H^*(B;\mathbb{Z})$-module. Since we are using $\mathbb{Z}$ coefficients instead of $\mathbb{Z}/2$ signs do matter now. We modify the defining relation for the Chern classes to be

$$x^n - c_1(E)x^{n-1} + \cdots + (-1)^n c_n(E) = 0$$

with alternating signs. The sign change does not affect the proofs of properties a)-c). For d), the sign convention turns the defining relation of $c_1(\gamma^1)$ into

$$x - c_1(\gamma^1) = 0$$

with $x = \alpha$. Thus $c_1(\gamma^1)$ is the chosen generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ (and not minus the generator). □

Proposition 16.4. *Regarding an n-dimensional complex vector bundle $E \to B$ as a $2n$-dimensional real vector bundle, then $w_{2i+1}(E) = 0$ and $w_{2i}(E)$ is the image of $c_i(E)$ under the homomorphism $H^{2i}(B;\mathbb{Z}) \to H^{2i}(B;\mathbb{Z}/2)$.*

Proof. There is a natural map $p\colon \mathbb{R}\mathrm{P}(E) \to \mathbb{C}\mathrm{P}(E)$ sending a real line to the complex line containing it. This projection fits into a commutative diagram

$$\begin{array}{ccccc}
\mathbb{R}\mathrm{P}^{2n-1} & \longrightarrow & \mathbb{R}\mathrm{P}(E) & \xrightarrow{\mathbb{R}\mathrm{P}(g)} & \mathbb{R}\mathrm{P}^\infty \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}(E) & \xrightarrow{\mathbb{C}\mathrm{P}(g)} & \mathbb{C}\mathrm{P}^\infty
\end{array}$$

where the left vertical map is the restriction of p to a fiber of E and the maps $\mathbb{R}\mathrm{P}(g)$ and $\mathbb{C}\mathrm{P}(g)$ are the projectivizations of a map $g\colon E \to \mathbb{C}^\infty$ which is injective and $\mathbb{C}$-linear on the fibers of E. All three vertical maps are fiber bundles with fiber $\mathbb{R}\mathrm{P}^1$, the real lines in a complex line (using $\mathbb{C} \cong \mathbb{R}$). The Leray-Hirsch theorem applies to the bundle $\mathbb{R}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$, so if α is the generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$, the $\mathbb{Z}/2$-reduction $\bar{\alpha} \in H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}/2)$ pulls back to a generator of $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$. This generator is β^2, the square of the generator $\beta \in H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$. Hence the $\mathbb{Z}/2$-reduction

$$\bar{x}_{\mathbb{C}}(E) = \mathbb{C}\mathrm{P}(g)^*(\bar{\alpha}) \in H^2(\mathbb{C}\mathrm{P}(E);\mathbb{Z}/2)$$

of the class $x_{\mathbb{C}}(E) = \mathbb{C}\mathrm{P}(g)^*(\alpha)$ pulls back to the square of the class

$$x_{\mathbb{R}}(E) = \mathbb{R}\mathrm{P}(g)^*(\alpha) \in H^1(\mathbb{R}\mathrm{P}(E); \mathbb{Z}/2).$$

Thus the $\mathbb{Z}/2$-reduction of the defining relation for the Chern classes of E, which is

$$\bar{x}_{\mathbb{C}}(E)^n + \bar{c}_1(E)\bar{x}_{\mathbb{C}}(E)^{n-1} + \cdots + \bar{c}_n(E) = 0,$$

(signs do not matter here since we are over $\mathbb{Z}/2$) pulls back to the relation

$$x_{\mathbb{R}}(E)^{2n} + \bar{c}_1(E)x_{\mathbb{R}}(E)^{2(n-1)} + \cdots + \bar{c}_n(E) = 0,$$

which is the defining relation for the Stiefel–Whitney classes of E. Hence we must have

$$w_{2i+1}(E) = 0 \text{ and } w_{2i}(E) = \bar{c}_i(E).$$

□

16.3. **The complex Grassmannian and its cohomology.** The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{n+k})$ is the space of complex k-planes in $\mathbb{C}^{n+k}$. We can topologize this space just as in the real case and we obtain a complex manifold of complex dimension kn or real dimension $2kn$. For $k = 1$, we get $\mathrm{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{C}\mathrm{P}^n$.

Moreover, the inclusions $\mathbb{C}^{n+k} \subset \mathbb{C}^{n+1+k} \subset \ldots$ induce inclusions

$$\mathrm{Gr}_k(\mathbb{C}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{C}^{n+1+k}) \subset \ldots$$

The infinite complex Grassmannian manifold is the union

$$\mathrm{Gr}_k(\mathbb{C}) := \mathrm{Gr}(\mathbb{C}^\infty) = \bigcup_n \mathrm{Gr}_k(\mathbb{C}^{n+k}).$$

This is the set of all k-dimensional complex vector subspaces of $\mathbb{C}^\infty$. The topology of $\mathrm{Gr}_k(\mathbb{C})$ is the direct limit topology. We have $\mathrm{Gr}_1(\mathbb{C}) = \mathbb{C}\mathrm{P}^\infty$.

The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{n+k})$ is equipped with a canonical k-dimensional complex vector bundle $\gamma^k(\mathbb{C}^{n+k})$ defined as in the real case. The total space

$$E = E(\gamma^k(\mathbb{C}^{n+k}))$$

is the set of all pairs

$$(\text{complex } k\text{-plane in } \mathbb{C}^{n+k}, \text{ vector in that } k\text{-plane}).$$

The topology on E is the topology as a subset of $\mathrm{Gr}_k(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k}$. The projection map

$$\pi\colon E \to \mathrm{Gr}_k(\mathbb{C}^{n+k}), \text{ is defined by } \pi(V, v) = V,$$

and the vector space structure is defined by

$$t_1(V, v_1) + t_2(V, v_2) = (V, t_1v_1 + t_2v_2).$$

Over the infinite complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$, there is also a canonical bundle $\gamma^k_{\mathbb{C}}$ whose total space is

$$E(\gamma^k_{\mathbb{C}}) \subset \mathrm{Gr}_k(\mathbb{C}) \times \mathbb{C}^\infty$$

the set of all pairs

$$(\text{complex } k\text{-plane in } \mathbb{C}^\infty, \text{ vector in that } k\text{-plane})$$

topologized as a subset of the product $\mathrm{Gr}_k(\mathbb{C}) \times \mathbb{C}^\infty$. The projection

$$\pi \colon E(\gamma^k_{\mathbb{C}}) \to \mathrm{Gr}_k(\mathbb{C})$$

is given by $\pi(V, v) = V$.

The crucial result is again the following theorem.

Theorem 16.5. *For a paracompact space B, the map $[B, \mathrm{Gr}_k(\mathbb{C})] \to \mathrm{Vect}^k_{\mathbb{C}}(B)$, $[f] \mapsto f^*(\gamma^k)$, is a bijection from the set of homotopy classes of maps $B \to \mathrm{Gr}_k(\mathbb{C})$ and the set of isomorphism classes of k-dimensional complex vector bundles.*

The proof is the same as for real bundles. The theorem justifies to call the infinite complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$ the *classifying space* and $\gamma^k_{\mathbb{C}}$ the *universal bundle* for k-dimensional complex vector bundles.

The complex Grassmannian $\mathrm{Gr}_k(\mathbb{C})$ is a CW-complex with one cell of dimension $2n$ corresponding to each partition of n into at most k integers.

Theorem 16.6. *The cohomology ring $H^*(\mathrm{Gr}_k(\mathbb{C}); \mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}$ freely generated by the Chern classes $c_1(\gamma^k_{\mathbb{C}}), \ldots, c_k(\gamma^k_{\mathbb{C}})$.*

Proof. Just work out the proof for the real Grassmannian in the complex case. □

17. Complex K-theory

From now on all vector bundles will complex vector bundles. For most of our arguments we will assume that the spaces are compact Hausdorff even though some statements may be true for more general spaces. In the following lectures we will mostly follow Atiyah's lecture notes on K-theory.

17.1. **Some basic definitions.** Let X be a space and let $\mathrm{Vect}(X)$ be the set of isomorphism classes of finite dimensional complex vector bundles. The set $\mathrm{Vect}(X)$ has the structure of an abelian semigroup under the composition of taking direct sums. We know that to any abelian semigroup A there is an associated abelian group $K(A)$ with the following universal property:

There is a semigroup homomorphism $\alpha\colon A \to K(A)$ such that if G is any group and $\gamma\colon A \to G$ any semigroup homomorphism, there is a unique homomorphism of groups $\kappa\colon K(A) \to G$ such that $\gamma = \kappa\alpha$. This determines $K(A)$ up to unique isomorphism.

There are different ways to construct $K(A)$. One way is to define $K(A)$ to be the set of pairs (a,b) in $A \times A$ modulo the following equivalence relation:

(7) $\quad (a,b) \sim (a',b')$ if there is a $c \in A$ such that $a + b' + c = a' + b + c$.

In other words,

$$K(A) = A \times A/\Delta(A),$$

where $\Delta\colon A \to A \times A$ denotes the diagonal.

Denoting the equivalence class of (a,b) by $[a,b]$ we can define the addition on $K(A)$ by

$$[a,b] + [a',b'] = [a + a',b + b']$$

The homomorphism $\alpha_A\colon A \to K(A)$ is defined by

$$a \mapsto [a,0],$$

where 0 denotes the zero element of A (which we assume to exist). The nice feature of this description of $K(A)$ is that the interchange of factors in $A \times A$ induces an inverse in $K(A)$ which makes $K(A)$ into a group.

The pair $(K(A), \alpha_A)$ is a functor of A so that if $f\colon A \to B$ is a semigroup homomorphism we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & K(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle K(f)} \\ B & \xrightarrow{\alpha_B} & K(B). \end{array}$$

Moreover, if B is a group then α_B is an isomorphism. This shows that $K(A)$ has the required universal property. Furthermore, if A is also a semiring, i.e., A is a semigroup with a multiplication that is distributive over the addition of A, then $K(A)$ is a ring with multiplication

$$[a,b] \cdot [a',b'] = aa' + bb', ab' + ba'].$$

Now if X is a space, we write $K(X)$ for the ring $K(\mathrm{Vect}(X))$, where the multiplication is given by forming tensor products of vector bundles. For $E \in \mathrm{Vect}(X)$ we will write $[E]$ for its image in $K(X)$, or also just E if there is no danger of confusion.

Before we proceed we need the following lemma.

Lemma 17.1. *Let B be a compact Hausdorff space and $\pi\colon E \to B$ be a complex vector bundle. Then there exists a complex vector bundle E' such that $E \oplus E'$ is a trivial bundle.*

Proof. From the case of real vector bundles, we know how to construct a map $g\colon E \to \mathbb{C}^\infty$ which is linear and injective on each fiber of π when B is paracompact. Since we assume here that B is compact, the construction of g shows that there is a some finite dimension N such that g factors through $E \to \mathbb{C}^N$.

This gives us a map $f\colon E \to B \times \mathbb{C}^N$. The image of f is a sub-bundle of $B \times \mathbb{C}^N$. Hence $E \to B$ is isomorphic to a sub-bundle of the trivial bundle $B \times \mathbb{C}^N$. The canonical Hermitian metric on this trivial bundle then yields a complementary sub-bundle E' such that $E \oplus E'$ is a trivial bundle. □

Our explicit description of $K(X)$ shows that every element of $K(X)$ is of the form $[E]-[F]$, where E and F are bundles over X. By the lemma, we can choose a bundle G such that $F \oplus G \cong \epsilon^n$ is a trivial bundle for some n. Then we have

$$[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\epsilon^n].$$

Thus, every element of $K(X)$ is of the form $[H] - [\epsilon^n]$.

Suppose now that E, F are such that $[E] = [F]$ in $K(X)$. Our explicit description (7) of $K(X)$ then shows that there is a bundle G such that $E \oplus G \cong F \oplus G$. Let G' be a bundle such that $G \oplus G' \cong \epsilon^n$. Then

$$E \oplus G \oplus G' \cong F \oplus G \oplus G', \text{ so } E \oplus \epsilon^n \cong F \oplus \epsilon^n.$$

We say that two bundles are *stably equivalent*, if they become isomorphic after adding suitable trivial bundles to them. The above argument then shows:

Lemma 17.2. *We have $[E] = [F]$ in $K(X)$ if and only if E and F are stably equivalent.*

Now suppose that $f\colon X \to Y$ is a continuous map. Then

$$f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

induces a ring homomorphism

$$f^*\colon K(Y) \to K(X).$$

By one of the problems on Problem Set 2, this homomorphism depends only on the homotopy class of f.

17.2. **The periodicity theorem.** The fundamental theorem for K-theory is the periodicity theorem. It says, in particular, that for any X, there is an isomorphism between $K(X) \otimes K(S^2)$ and $K(X \times S^2)$. We will prove actually prove a more general statement which we will now explain.

Let E be a vector bundle over a space X, and let $\mathbb{P}(E)$ be the projective bundle (of complex lines) over X associated to E. If $p\colon \mathbb{P}(E) \to X$ is the projection map, $p^{-1}(x)$ is a complex projective space for all $x \in X$.

Remark 17.3. Projective spaces and bundles have the following nice property:

If V is a (complex) vector space, and W is a vector space of dimension one, then V and $V \otimes W$ are isomorphic, but not naturally isomorphic. However, taking projective spaces makes things easier.

For any non-zero element $w \in W$ the map

$$v \mapsto v \otimes w$$

defines an isomorphism between V and $V \otimes W$, and thus defines an isomorphism

$$\mathbb{P}(w)\colon \mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}(V \otimes W).$$

However, if w' is any other non-zero element of W, $w' = \lambda w$ for some non-zero complex number $\lambda \in \mathbb{C}^*$. Thus

$$\mathbb{P}(w) = \mathbb{P}(w'),$$

so the isomorphism between $\mathbb{P}(V)$ and $\mathbb{P}(V \otimes W)$ is natural.

Thus, if E is any vector bundle, and L is a line bundle, there is a natural isomorphism

$$\mathbb{P}(E) \cong \mathbb{P}(E \otimes L),$$

which concludes our remark.

If E is any vector bundle over X then each point $a \in \mathbb{P}(E)_x = \mathbb{P}(E_x)$ represents a one-dimensional subspace $H_x^* \subset E_x$. The union of all these defines a subspace

$$H^* \subset p^*E,$$

which consists of pairs of one-dimensional subspace in a fiber and a point on that line.

Lemma 17.4. *The space H^* is a sub-bundle of p^*E over $\mathbb{P}(E)$.*

Proof. The problem is local, so we may assume that E is a trivial. Then the lemma reduces to the fact that the canonical line bundle over $\mathbb{C}\mathrm{P}^n$ is a sub-bundle of the pullback of a trivial bundle. □

Remark 17.5. Note that we have met the real version of this line bundle before when we proved the splitting principle.

Definition 17.6. Now we define H to be the dual line bundle of H over $\mathbb{P}(E)$, i.e., for $\epsilon := \epsilon^1_{\mathbb{P}(E)}$ the trivial line bundle over $\mathbb{P}(E)$,

$$H := \mathrm{Hom}(H^*, \epsilon)$$

Remark 17.7. The choice of using H instead of H^* has historical reasons and is related to the use of canonical line- and quotient bundles in algebraic geometry. We will come back to this point later.

Example 17.8. Let X be compact space and let $\epsilon \oplus \epsilon$ be the sum of two trivial line bundles over X. Then

$$\mathbb{P}(\epsilon \oplus \epsilon) \cong X \times S^2,$$

since the bundle $\epsilon \oplus \epsilon$ has total space $X \times \mathbb{C}^2$, and hence

$$\mathbb{P}(\epsilon \oplus \epsilon) \cong X \times \mathbb{C}\mathrm{P}^1 \cong X \times S^2.$$

Moreover, H^* is just the pullback of the canonical complex line bundle $\gamma^1_{\mathbb{C}}$ over $\mathbb{C}\mathrm{P}^1$ to $X \times \mathbb{C}\mathrm{P}^1 \cong X \times S^2$. Hence H is the dual line bundle

$$H = \mathrm{Hom}(\gamma^1_{\mathbb{C}}, \epsilon).$$

We can now state the periodicity theorem.

Theorem 17.9. *Let X be a compact space, let L be a line bundle over X, and let $H = H(L \oplus \epsilon)$. Then, as a $K(X)$-algebra, $K(\mathbb{P}(L \oplus \epsilon))$ is generated by $[H]$, and is subject to the single relation*

$$([H] - [\epsilon])([L][H] - [\epsilon]) = 0.$$

The proof will be the topic of following lectures. Today we just point out two consequences of the theorem. The first one follows from the theorem and Example 17.8 for $X = *$ a point (and $L = \epsilon$ the trivial line bundle).

Corollary 17.10. *As a $K(*)$-module $K(S^2)$ is generated by $[H]$ and $[H]$ is subject to the single relation*

$$([H] - [\epsilon])^2 = 0.$$

The second one requires a little bit of analysis of the ring structures given by Theorem 17.9 and Corollary 17.10.

Corollary 17.11. *Let X be a compact space and*

$$\mu\colon K(X) \otimes K(S^2) \to K(X \times S^2)$$

be defined by

$$\mu(a \otimes b) = (\pi_1^* a)(\pi_2^* b),$$

where π_1 and π_2 are the projections onto the two factors. Then μ is an isomorphism of rings.

Proof. We know from Example 17.8

$$\mathbb{P}(\epsilon_X \oplus \epsilon_X) \cong X \times S^2 \text{ and } \mathbb{P}(\epsilon_* \oplus \epsilon_*) \cong S^2.$$

Under the canonical map $\pi_2\colon X \times S^2 \to * \times S^2$, the class $[H_*] \in K(S^2)$ is pulled back to the class $[H_X] \in K(X \times S^2)$. Using Theorem 17.9, we see that μ becomes the homomorphism

$$K(X) \otimes_{K(*)} K(*)[[H_*]]/(([H_*] - 1)^2) \to K(X)[[H_X]]/(([H_X] - 1)^2)$$

which by the above is just

$$K(X) \otimes_{K(*)} K(*)[[H_*]]/(([H_*] - 1)^2) \to K(X)[\pi_2^*([H_*])]/((\pi_2^*([H_*)] - 1)^2)$$

which is an isomorphism of rings. □

18. Complex K-theory as a representable functor

We postpone the proof of the periodicity theorem for a while and first workout more properties of the K-theory functor.

18.1. **Reduced K-theory.** Let X be a compact Hausdorff space. Recall that a vector bundle over X may have different dimensions on the connected components of X. If X is a based space, i.e., has a chosen base point $* \in X$, then we can define a function

$$d\colon \mathrm{Vect}(X) \to \mathbb{Z}$$

that sends a vector bundle to the dimension of its restriction to the component of the basepoint $*$. The function d is a homomorphism of semirings and hence induces a dimension function

$$d\colon K(X) \to \mathbb{Z},$$

which is a homomorphism of rings. Since d is an isomorphism when X is a point, d can be identified with the induced map

$$K(X) \to K(*).$$

This leads to the following definition.

Definition 18.1. The *reduced K-theory* $\tilde{K}(X)$ of a based space is the kernel of $d\colon K(X) \to \mathbb{Z}$.

Remark 18.2. $\tilde{K}(X)$ is an ideal of $\mathrm{K}(X)$ and thus a ring without identity. It clearly holds

$$K(X) \cong \tilde{K}(X) \times \mathbb{Z}.$$

If X does not have a base point yet, let

$$X_+ := X \amalg *$$

be X together with a disjoint base point. Then we have

$$K(X) \cong \tilde{K}(X_+).$$

We denote the stable equivalence class of a bundle ξ by $\{\xi\}$ and the set of stable equivalence classes of finite dimensional complex vector bundles over X by $EU(X)$. The set $EU(X)$ forms an abelian group under direct sums, since we know that for each bundle ξ there is bundle ξ' such that $\xi \oplus \xi'$ is trivial.

Proposition 18.3. *There is a natural isomorphism of groups $EU(X) \xrightarrow{\cong} \tilde{K}(X)$.*

Proof. Denote the class of the trivial n-dimensional bundle ϵ^n over X by n. Then we know that every element in $K(X)$ can be written in the form $[\xi] - q$ for some vector bundle ξ and some non-negative integer q. Then we can define the required homomorphism by

$$\{\xi\} \mapsto [\xi] - d(\xi).$$

It is clear that this map is surjective and it is injective, since we know from the previous lecture that $[\xi] = [\eta]$ if and only if $\{\xi\} = \{\eta\}$. □

18.2. **Complex K-theory as a representable functor.** Let $\mathrm{Gr}_n(\mathbb{C})$ be the infinite dimensional complex Grassmannian manifold of complex n-planes. It is also common to write

$$BU(n) := \mathrm{Gr}_n(\mathbb{C}).$$

We know from Lecture 16 that there is a natural bijection

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, BU(n)]$$

where $[-,-]$ denotes homotopy classes of maps. As we have just seen base points can play a role for studying K-theory (as for any other cohomology theory). Let $[-,-]_*$ denote the set of homotopy classes of basepoint preserving maps. Then we have

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X_+, BU(n)]_*.$$

The map $V \mapsto \mathbb{C} \oplus V$ defines an inclusion

$$i_n \colon BU(n) \to BU(n+1),$$

and we denote the colimit by

$$BU := \operatorname*{colim}_n BU(n)$$

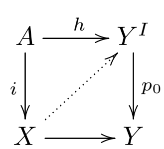
with the direct limit (or union) topology.

Recall that a space is *nondegenerately based*, or *well-pointed*, if the inclusion of its basepoint is a cofibration.[1]

Theorem 18.4. *We endow $\mathbb{Z}$ with the discrete topology. For any compact space X, there is a natural isomorphism*

$$K(X) \cong [X_+, BU \times \mathbb{Z}]_*.$$

[1] A map $i \colon A \to X$ is a cofibration if for any commutative diagram of the form



there exists an $\tilde{h} \colon X \to Y^I$ that makes the diagram commute.

For a nondegenerately based compact space X, there is a natural isomorphism

$$\tilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*.$$

Proof. When X is connected and ξ is an n-dimensional bundle over X with associated classifying map

$$f_\xi \colon X \to BU(n) \subset BU,$$

the first isomorphism sends

$$[\xi] - q \text{ to the pair } (f_\xi, n - q).$$

(Note that since $\mathbb{Z}$ is discrete, the map $X \to \mathbb{Z}$ must be constant.) Then we obtain also an isomorphism for non-connected spaces since both functors $K(-)$ and $[-, BU \times \mathbb{Z}]_*$ send disjoint unions to cartesian products.

For the second isomorphism follows from the first. For let $S^0 \to X_+$ be the cofibration induced by the basepoint and the disjoint basepoint. Then we can identify $d \colon K(X) \to \mathbb{Z}$ with the induced map

$$[X_+, BU \times \mathbb{Z}]_* \to [S^0, BU \times \mathbb{Z}]_*.$$

Hence we need to show that the kernel of this map is $[X, BU \times \mathbb{Z}]_*$. The cofibration $S^0 \to X_+$ with $X_+/S^0 = X$ induces an exact sequence

$$[S^1 \wedge S^0, BU \times \mathbb{Z}]_* \to [X, BU \times \mathbb{Z}]_* \to [X_+, BU \times \mathbb{Z}]_* \to [S^0, BU \times \mathbb{Z}]_*.$$

The left hand set is equal to $[S^1, BU \times \mathbb{Z}]_*$. Since we are looking at basepoint preserving maps, this is just $[S^1, BU]_+ = \pi_1(BU)$. Hence we need to show that $\pi_1(BU)$ is trivial or in other words that BU is simply connected. But $\pi_1(BU)$ is isomorphic to the set of isomorphism classes of complex vector bundles over S^1. We will show on the next problem set that this set is trivial. □

For more general, non-compact, spaces it is best to define K-theory to be the functor represented by the space $BU \times \mathbb{Z}$.

Definition 18.5. For a space X of the homotopy type of a CW-complex, we define

$$K(X) := [X_+, BU \times \mathbb{Z}]_*.$$

For a nondegenerately based space X of the homotopy type of a CW-complex, we define

$$\tilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*.$$

When X is compact, we know that $K(X)$ is a ring. The following result shows that is also true for more general spaces.

Proposition 18.6. *The space $BU \times \mathbb{Z}$ is a ring space up to homotopy. This means that there are additive and multiplicative structures on $BU \times \mathbb{Z}$ such that the associativity, commutativity, and distributivity diagrams required of a ring commute up to homotopy.*

Idea of the proof. For the additive structure, note that taking direct sums induces maps for each m and n

$$\mathrm{Gr}_m(\mathbb{C}^\infty) \times \mathrm{Gr}_n(\mathbb{C}^\infty) \to \mathrm{Gr}_{m+n}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty.$$

After choosing an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$ we get a map

$$BU(m) \times BU(n) \to BU(m+n).$$

Taking colimits over m and n then yields a map

$$\oplus \colon BU \times BU \to BU.$$

This map is associative and commutative up to homotopy. The zero-dimensional plane provides a basepoint which is a zero element up to homotopy. Using the ordinary addition on $\mathbb{Z}$, we obtain the additive H-space structure on $BU \times \mathbb{Z}$. For multiplication, taking the tensor product of the canonical bundles induces a homotopy class of classifying maps

$$BU(m) \times BU(n) \to BU(mn).$$

With a lot more effort than for direct sums, one can show that these maps pass to colimits and define a multiplicative H-space structure on $BU \times \mathbb{Z}$. □

19. Complex K-theory as a cohomology theory

19.1. **K-theory as a cohomology theory.** Let $\mathcal{C}$ be the category of compact Hausdorff spaces, $\mathcal{C}^+$ be the category of compact Hausdorff spaces with a distinguished basepoint, and $\mathcal{C}^2$ the category of pairs. We have defined K-theory as functors K on $\mathcal{C}$ and $\tilde{K}$ on $\mathcal{C}^+$. We extend it a functor on $\mathcal{C}^2$ by defining

$$K(X,Y) := \tilde{K}(X/Y)$$

for any pair of compact spaces (X,Y).

Definition 19.1. For $n \geq 0$, we define functors by

$$\begin{array}{lcll} \tilde{K}^{-n}(X) & = & \tilde{K}(S^n X) = \tilde{K}(S^n \wedge X) & \text{for } X \in \mathcal{C}^+ \\ K^{-n}(X,Y) & = & \tilde{K}^{-n}(X/Y) = \tilde{K}(S^n(X/Y)) & \text{for } (X,Y) \in \mathcal{C}^2 \\ K^{-n}(X) & = & \tilde{K}^{-n}(X,\emptyset) = \tilde{K}(S^n(X_+)) & \text{for } X \in \mathcal{C} \end{array}$$

which are contravariant on the appropriate categories.

Lemma 19.2. *For $(X,Y) \in \mathcal{C}^2$ we have an exact sequence*

$$K(X,Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y)$$

where $i \colon Y \to X$ and $j \colon (X,\emptyset \to (X,Y)$ are the inclusions.

Proof. We could apply the representability of K-theory of the previous lecture. But there is a very nice direct way to prove the lemma:
The composition i^*j^* is induced by the composition

$$j \circ i \colon (Y,\emptyset) \to (X,Y)$$

and so factors through the zero group $K(Y,Y)$. Thus $i^*j^* = 0$. Suppose now that $\alpha \in \mathrm{Ker}\,(i^*)$. We may represent α in the form $[\xi] - n$ where ξ is a vector bundle over X. Since $i^*(\alpha) = 0$ it follows that

$$[\xi|Y] = n \text{ in } K(Y).$$

This implies that for some integer m we have

$$(\xi \oplus \epsilon^m)|Y = \epsilon^n \oplus \epsilon^m,$$

i.e., we have a trivialization h of $(\xi \oplus \epsilon^m)|Y$. This defines a bundle $(\xi \oplus \epsilon^m)/h$ on X/Y in the following way. The total space is the quotient of the total space of $\xi \oplus \epsilon^m$ modulo the relation

$$h^{-1}(y,v) \sim h^{-1}(y',v) \text{ for } y, y' \in Y,$$

and the projection is just the induced quotient map. We omit the details to show that this projection map staisfies local triviality. So we can define an element

$$\alpha' = [(\xi \oplus \epsilon^m)/h] - [\epsilon^n \oplus \epsilon^m] \in \tilde{K}(X/Y) = K(X,Y).$$

Then

$$\begin{aligned} j^*(\alpha') &= [\xi \oplus \epsilon^m] - [\epsilon^n \oplus \epsilon^m] \\ &= [E] - n = \xi. \end{aligned}$$

Thus α is in the image of j^* and we have $\mathrm{Ker}\,(i^*) = \mathrm{Im}(j^*)$, which proves the exactness. □

Corollary 19.3. *For $(X,Y) \in \mathcal{C}^2$ and $Y \in \mathcal{C}^+$ (hence $X \in \mathcal{C}^+$ by taking the same basepoint $y_0 \in X$) the sequence*

$$K(X,Y) \xrightarrow{i^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(Y)$$

is exact.

Proof. This follows from the previous lemma and the natural isomorphisms

$$K(X) \cong \tilde{K}(X) \oplus K(y_0)$$

and

$$K(Y) \cong \tilde{K}(Y) \oplus K(y_0).$$

□

Proposition 19.4. *For $(X,Y) \in \mathcal{C}^2$ there is a natural exact sequence which extends infinitely to the left*

$$\cdots \to K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y).$$

Proof. it suffices to show the exactness only for the sequence with terms of degree -1 and 0. Once we have done that we cann apply suspensions and extend the sequence to the left.

Let C and S denote cone and suspension respectively. Then we the following sequence of maps

$$\begin{array}{ccccccccc} Y & \hookrightarrow & X & \hookrightarrow & X \cup CY & \hookrightarrow & (X \cup CY) \cup CX & \hookrightarrow & ((X \cup CY) \cup CX) \cup C(X \cup CY) \\ & & & & \downarrow^{p} & & \downarrow & & \downarrow \\ & & & & X/Y & & SY & & SX \end{array}$$

The vertical maps are the quotient maps obtained by collapsing the most recently attached cone to a point. Now we successicely apply Corollary 19.3 to the pairs $(X \cup CY, X)$, $((C \cup CY) \cup (CX), X \cup CY)$, and $(((X \cup CY) \cup CX), ((X \cup CY) \cup CX) \cup C(X \cup CY))$. We start with the pair $(X \cup CY, X)$. By Corollary 19.3 we get an exact sequence

$$K(X \cup CY, X) \xrightarrow{m^*} \tilde{K}(X \cup CY) \xrightarrow{k^*} \tilde{K}(X).$$

Since CY is contractible, this implies by Lemma 19.6 below that

$$p^* \colon \tilde{K}(X/Y) \to \tilde{K}(X \cup CY)$$

is an isomorphism. The composition k^*p^* coincides with j^*. Let

$$\theta\colon K(X \cup CY, X) \to K^{-1}(Y) = K(S^1 \wedge Y_+)$$

be the isomorphism induced by the homeomorphisms

$$(X \cup CY)/X \approx CY/Y \approx S^1 \wedge Y_+.$$

Then defining

$$\delta\colon K^{-1}(Y) \to K(X,Y) \text{ by } \delta = m^*\theta^{-1}$$

we obtain a diagram

$$\begin{array}{ccccc}
\tilde{K}^{-1}(Y) & \xrightarrow{\quad\delta\quad} & K(X,Y) & \xrightarrow{\;j^*\;} & \tilde{K}(X) \\
\downarrow{\scriptstyle \theta^{-1}} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle =} \\
K(X \cup CY, X) & \xrightarrow{\;m^*\;} & \tilde{K}(X \cup CY) & \xrightarrow{\;k^*\;} & \tilde{K}(X)
\end{array}$$

where the vertical maps are isomorphisms/identities. Hence we obtain the exact sequence

$$\tilde{K}^{-1}(Y) \xrightarrow{\delta} K(X,Y) \xrightarrow{j^*} \tilde{K}(X).$$

Applying the same sort of arguments to the remaining pairs yields the remaining exactness (though it is a bit more complicated than the previous case). □

Example 19.5. In particular, we see that if X is the wedge sum $A \vee B$, then $X/A = B$ and the sequence breaks up into split short exact sequences. This implies

$$\tilde{K}(X) \cong \tilde{K}(A) \oplus \tilde{K}(B).$$

Lemma 19.6. *Let $Y \subset X$ be closed contractible subspace. Then the quotient map $q\colon X \to X/Y$ induces a bijection*

$$q^*\colon \mathrm{Vect}_{\mathbb{C}}(X/Y) \to \mathrm{Vect}_{\mathbb{C}}(X).$$

Proof. Let $p\colon E \to X$ be a bundle over X. Since Y is contractible, $E|Y$ is trivial. Thus there is a trivialization h

$$h\colon E|Y \to Y \times \mathbb{C}^n.$$

Moreover, two such trivializations differ by an automorphism of $Y \times \mathbb{C}^n$, i.e., by a map $Y \to \mathrm{GL}_n(\mathbb{C})$. But $\mathrm{GL}_n(\mathbb{C})$ is connected and V is contractible. Thus h is unique up to homotopy and so the isomorphism class of E/h is uiquely determined by that of E. Thus we have constructed a map

$$\mathrm{Vect}_{\mathbb{C}}(X) \to \mathrm{Vect}_{\mathbb{C}}(X/Y)$$

and this is a two-sided inverse for q^*. □

This shows that the complex K-theory functor behaves very much like the singular cohomology functor. In fact, complex K-theory defines a complex oriented cohomology theory.

19.2. **Bott periodicity for $\tilde{K}$.** We want a version of the periodicity theorem for the reduced groups too. We start with the following observation.

Lemma 19.7. *For nondegenerately based spaces X and Y, the projections of $X \times Y$ on X and Y and the quotient map $X \times Y \to X \wedge Y$ induce a natural isomorphism*

$$\tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \cong \tilde{K}(X \times Y).$$

The group $\tilde{K}(X \wedge Y)$ is the kernel of the map

$$\tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$$

induced by the inclusions of X and Y into $X \times Y$.

Proof. The inclusions and projections make X and Y into retracts of $X \times Y$. This implies that the map

$$\tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$$

induced by the inclusions is a split surjection with splitting

$$\tilde{K}(X) \oplus \tilde{K}(Y) \to \tilde{K}(X \times Y),\ (a,b) \mapsto p_1^*(a) + p_2^*(b)$$

where p_1 and p_2 are the projections. The inclusion $X \vee Y \to X \times Y$ is a cofibration by our assumption on X and Y. The quotient of this map is $X \wedge Y$. This cofibration induces an exact sequence

$$\tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y).$$

Since we have

$$\tilde{K}(X \vee Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y)$$

this proves the lemma. □

Lemma 19.8. *The Künneth map*

$$\mu \colon K(X) \otimes K(Y) \to K(X \times Y)$$

defined by

$$\mu(a \otimes b) = (p_1^* a)(p_2^* b),$$

where p_1 and p_2 are the projections onto the two factors, induces a reduced map

$$\tilde{\mu} \colon \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y).$$

Proof. For: Let $x_0 \in X$ and $y_0 \in Y$ be the basepoints, and let $a \in \tilde{K}(X) = \mathrm{Ker}(K(X) \to K(x_0))$ and $b \in \tilde{K}(Y) = \mathrm{Ker}(K(Y) \to K(y_0))$. Then $p_1^* a$ restricts to zero in $K(Y)$ and $p_2^* b$ restricts to zero in $K(X)$. Hence the product $(p_1^* a)(p_2^* b) \in K(X \times Y)$ restricts to zero in both $K(X)$ and $K(Y)$ and hence in $K(X \vee Y)$. In particular, $(p_1^* a)(p_2^* b)$ lies in $\tilde{K}(X \times Y)$. Now Lemma 19.7 implies that $(p_1^* a)(p_2^* b)$ pulls back to a unique element in $\tilde{K}(X \wedge Y)$. This defines the reduced Künneth map $\tilde{\mu}$. □

We have a reduced splitting

$$K(X) \otimes K(Y) \cong \tilde{K}(X) \otimes \tilde{K}(Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z},$$

which is compatible with the splitting of Lemma 19.7 and shows that the reduced Künneth map is a ring homomorphism.

The unreduced version of the periodicity theorem of the previous lecture now implies the following reduced version.

Theorem 19.9. *For nondegenerately based compact spaces X, the map*

$$\tilde{\mu}\colon \tilde{K}(X) \otimes \tilde{K}(S^2) \to \tilde{K}(X \wedge S^2)$$

is an isomorphism.

Let H^* be the canonical line bundle over $\mathbb{C}\mathrm{P}^1 = S^2$ and H be its dual. We know from the previous lecture

$$K(S^2) \cong \mathbb{Z}[H]/(([H]-1)^2),$$

and hence

$$\tilde{K}(S^2) \text{ is the ideal } \mathbb{Z}([H]-1).$$

Then Theorem 19.9 implies the following version of Bott periodicity.

Theorem 19.10 (Bott periodicity)**.** *For nondegenerately based compact spaces X, the map*

$$\beta\colon \tilde{K}(X) \to \tilde{K}(X \wedge S^2),\ a \mapsto \tilde{\mu}(a, [H]-1)$$

is an isomorphism.

Corollary 19.11. *We have $\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) = \mathbb{Z}$, generated by the n-fold recuced product $([H]-1)^n$.*

20. K-theory of complex projective spaces

This was a guest lecture by Mike Hopkins. Unfortunately, there are no notes available.

21. Splitting principle and the projective bundle formula in K-theory

This was a guest lecture by Mike Hopkins. Unfortunately, there are no notes available.

22. Thom classes and the Thom isomorphism in K-theory

This was a guest lecture by Mike Hopkins. Unfortunately, there are no notes available.

23. Proof of the Periodicity Theorem I

We still need to prove the periodicity theorem for complex K-theory. We will prove it in the following special form. The proof of the more general form of Lecture 17 is very similar. Let X be a compact Hausdorff space and H the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$. We calculated ine one of the homework problems that we have the relation

$$(H \otimes H) \oplus 1 \cong H \oplus H,$$

or in other words, in $K(S^2)$ we have $(H^1 - 1) = 0$. This shows that there is a natural homomorphism of rings

$$\mathbb{Z}[H]/(H-1)^2 \to K(S^2).$$

Theorem 23.1. *The natural homomorphism*

$$\mu\colon K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$$

is an isomorphism of rings.

The proof of the theorem will occupy the rest of today's lecture and the next one. It is based on a careful analysis of the construction of complex vector bundles on $X \times S^2$ via clutching functions. In our exposition we follow Hatcher's notes. We encourage everyone to read Atiyah's original lecture notes as well.

23.1. **Clutching functions.** We saw on Problem Set 4 that isomorphism classes of complex vector bundles over S^2 correspond to homotopy classes of maps

$$S^1 \to \mathrm{GL}_n(\mathbb{C}).$$

Such functions are called *clutching functions*. In the proof of Theorem 32.1 we make use of this idea to construct vector bundles over $X \times S^2$.

Let $p\colon E \to X$ be a vector bundle and let $f\colon E \times S^1 \to E \times S^1$ be an automorphism of the product vector bundle

$$p \times \mathrm{id}\colon E \times S^1 \to X \times S^1.$$

This means that for each $x \in X$ and $z \in S^1$, f specifies an isomorphism

$$f(x,z)\colon p^{-1}(x) \to p^{-1}(x).$$

From E and f we construct a vector bundle over $X \times S^2$ by taking two copies of $E \times D^2$ and identifying the subspaces $E \times S^1$ via f. We write this bundle as $[E,f]$, and call f a *clutching function* for $[E,f]$. If

$$f_t\colon E \times S^1 \to E \times S^1$$

is a homotopy of clutching functions, then we get an induced isomorphism

$$[E, f_0] \cong [E, f_1]$$

since from the homotopy f_t we can construct a vector bundle over $X \times S^2 \times I$ restricting to $[E, f_0]$ and $[E, f_1]$ over $X \times S^2 \times \{0\}$ and $X \times S^2 \times \{1\}$. It is also clear from the definitions that

$$[E_1, f_1] \oplus [E_2, f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2].$$

Let us have a look at some examples:

Example 23.2. For the identity map on S^1, $[E, \mathrm{id}]$ is just the pullback of E via the projection $X \times S^2 \to X$. As an element in $K(X \times S^2)$, $[E, \mathrm{id}]$ is equal to $\mu(E \otimes 1)$.

Example 23.3. Recall the clutching function for the canonical line bundle H over $\mathbb{C}\mathrm{P}^1$: We can write the elements $[Z_0, z_1]$ of $\mathbb{C}\mathrm{P}^1$ as ratios

$$z = z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2.$$

Then we can write points in the disk D^2_0 inside the unit circle $S^1 \subset \mathbb{C}$ uniquely in the form

$$[z_0/z_1, 1] = [z, 1] \text{ with } |z| \leq 1,$$

and points in the disk D^2_∞ outside S^1 can be written uniquely in the form

$$[1, z_1/z_0] = [1, z^{-1}] \text{ with } |z^{-1}| \leq 1.$$

Over D^2_0 the map

$$[z, 1] \mapsto (z, 1)$$

defines a section of the canonical line bundle, and over D^2_∞ a section is

$$[1, z^{-1}] \mapsto (1, z^{-1}).$$

These sections determine trivializations of the canonical line bundle over these two disks, and over their common boundary S^1 we pass from the trivialization of D^2_∞ to the trivialization of D^2_0 by multiplying with z. Thus by taking D^2_∞ as D^2_+ and D^2_0 as D^2_- we see that the canonical line bundle has the clutching function

$$f \colon S^1 \to \mathrm{GL}_n(\mathbb{C}),\; f(z) = (z).$$

Example 23.4. a) Taking X to be a point in the previous example, we get

$$[1, z] \cong H,$$

where 1 is the trivial line bundle over the point and z means scalar multiplication by $z \in S^1 \subset \mathbb{C}$.

b) More generally, for $n \geq 0$ we have

$$[1, z^n] \cong H \otimes \cdots \otimes H = H^n.$$

Writing H^{-1} for the inverse of H with respect to the tensor product in $K(X)$, i.e., $H \otimes H^{-1} \cong 1$, we can extend this formula to negative n too. For $n \leq 0$, we have

$$[1, z^n] \cong H^{-1} \otimes \cdots \otimes H^{-1} = H^n.$$

Example 23.5. a) Now if E is a vector bundle over a compact space X, we deduce from the previous examples

$$[E,z^n] \cong \mu(E \otimes \hat{H}^n) \text{ for } n \in \mathbb{Z},$$

where $\hat{H}^n$ denotes the pullback of H^n via the projection $X \times S^2 \to S^2$.
b) More generally, if f is a clutching function we get

$$[E,z^n f] \cong [E,f] \otimes \hat{H}^n \text{ for } n \in \mathbb{Z}.$$

A key observation is that every bundle over $X \times S^2$ comes from a clutching function. More precisely:

Lemma 23.6. *Let $F \to X \times S^2$ be a vector bundle of dimension n. Then there is an n-dimensional bundle $E \to X$ and a clutching function $f\colon S^1 \to \mathrm{GL}_n(\mathbb{C})$ such that*

$$F \cong [E,f] \text{ over } X \times S^2.$$

Proof. As in Example 23.3, we consider the unit circle $S^1 \subset \mathbb{C} \cup \{\infty\} = S^2$ and decompose S^2 into the two disks D_0 and D_∞. Let F_α denote the restriction of F to $X \times D_\alpha$ for $\alpha = 0, \infty$. Now we define E to be the restriction of F to $X \times \{1\}$. Since D_α is a disk, the projection

$$X \times D_\alpha \to X \times \{1\}$$

is homotopic to the identity map of $X \times D_\alpha$, so the bundle F_α is isomorphic to the pullback of E by the projection map, and this pullback is $E \times D_\alpha$. This shows we have an isomorphism

$$h_\alpha\colon F_\alpha \to E \times D_\alpha.$$

Then we get

$$f = h_0 h_\infty^{-1} \text{ as a clutching function for } F.$$

□

Remark 23.7. We may assume that a clutching function f is *normalized* to be the identity over $X \times \{1\}$, since we may normalize any isomorphism of the form $h_\alpha\colon E_\alpha \to E \times D_\alpha$ by composing it over each $X \times \{z\}$ with the inverse of its restriction over $X \times \{1\}$.
Moreover, any two choices of normalized h_α are homotopic through normalized h_α's, since they differ by a map g_α from D_α to the automorphisms of E with $g_\alpha(1) = \mathrm{id}$, and such a g_α is homotopic to the constant map id by composing it with a deformation retraction of D_α to $*$.

Thus any two choices f_0 and f_1 of normalized clutching functions are joined by a homotopy of normalized clutching functions f_t.

We now know that clutching functions are a tool to understand all vector bundles over $X \times S^2$. The proof of Theorem 32.1 will require that we understand all possible clutching functions that are needed to construct all vector bundles over $X \times S^2$. The strategy will be to successively simplify the clutching functions.

23.2. **Laurent polynomial clutching functions.** The first step is to reduce to *Laurent polynomial* clutching functions, which have the form

$$\ell(x,z) = \sum_{|i|\leq n} a_i(x) z^i$$

where $a_i \colon E \to E$ is a map which restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. Such an a_i will be called an *endomorphism* of E.

Note: The linear transformation $a_i(x)$ is not required to be invertible, hence the terminology. Nevertheless, the linear combination $\sum_{|i|\leq n} a_i(x) z^i$ must be invertible, since clutching functions are automorphisms.

Hence the first step is to prove the following simplification.

Proposition 23.8. *Every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy*

$$\ell_t(x,z) \sum_{|i|\leq n} a_i(x,t) z^i.$$

The proof is based on the fact that on a compact space X, we can approximate continuous functions $f \colon X \times S^1 \to \mathbb{C}$ by Laurent polynomial functions of the form

$$\sum_{|n|\leq N} a_n(x) z^n = \sum_{|n|\leq N} a_n(x) e^{in\theta},$$

where $z = e^{i\theta} \in S^1$ and each a_n is a continuous function $X \to \mathbb{C}$. Motivated by Fourier series, we set

$$a_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x,e^{i\theta}) e^{-in\theta} d\theta.$$

For positive real r, consider the series

$$u(x,r,\theta) = \sum_{n\in\mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}.$$

For fixed $r < 1$, this series converges absolutely and uniformly as (x,θ) ranges over $X \times [0,2\pi]$. This follows from the fact that the geometric series

$$\sum_n r^n$$

converges, and, since $X \times S^1$ is compact,

$$|f(x, e^{i\theta})| \text{ is bounded and hence also } |a_n(x)|.$$

Now we need to show that $u(x,r,\theta)$ approaches $f(x,e^{i\theta})$ uniformly in x and θ as r goes to 1. For then sums of finitely many terms in the series for $u(r,x,\theta)$ with r near 1 will give the desired approximations to f by Laurent polynomial functions. Hence we need the following lemma.

Lemma 23.9. *As $r \to 1$, $u(r,x,\theta) \to f(x,e^{i\theta})$ uniformly in x and θ.*

Proof. For $r < 1$ we have

$$\begin{aligned} u(x,r,\theta) &= \textstyle\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} r^{|n|} e^{in(\theta - t)} f(x,e^{it}) dt \\ &= \textstyle\int_0^{2\pi} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} f(x,e^{it}) dt \end{aligned}$$

where the order of summation and integration can be interchanged since the series in the latter formula converges uniformly, by comparison with the geometric series $\sum_n r^n$. Define the Poisson kernel

$$P(r,\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\varphi} \text{ for } 0 \leq r \leq 1 \text{ and } \varphi \in \mathbb{R}.$$

Then we have

$$u(r,x,\theta) = \int_0^{2\pi} P(r, \theta - t) f(x,e^{it}) dt.$$

By summing the two geometric series for positive and negative n in the formula for $P(r, \varphi)$, one computes that

$$P(r,\varphi) = \frac{1}{2\pi}[1 - \frac{1}{1 - re^{i\varphi}} + \frac{1}{1-re^{-i\varphi}}] = \frac{1}{2\pi} \frac{1-r^2}{1 - 2r\cos\varphi + r^2},$$

where one uses the formula

$$e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi.$$

We will need three facts about $P(r,\varphi)$:

(a) As a function of φ, $P(r,\varphi)$ is even, of period 2π, and monotone decreasing on $[0,\pi]$, since the same is true for $\cos\varphi$ which appears in the denominator of $P(r,\varphi)$ with a minus sign. In particular, we have

$$P(r,\varphi) \geq P(r,\pi) > 0 \text{ for all } r < 1.$$

(b) $\int_0^{2\pi} P(r,\varphi)d\varphi = 1$ for each $r < 1$. This follows from integrating the series for

$$P(r,\varphi) = \frac{1}{2\pi}[1 + 2\sum_{n=1}^{\infty} \cos(n\varphi)]$$

term by term (the integral over all terms in the sum yield 0 and the integral over 1 yields 2π).

(c) For fixed $\varphi \in (0,\pi)$, $P(r,\varphi) \to 0$, since the numerator of $P(r,\varphi)$ approaches 0 and the denominator approaches $2 - 2\cos\varphi \neq 0$.

Now to show uniform convergence of $u(r,x,\theta)$ to $f(x,e^{i\theta})$ we first observe that, using (b), we have

$$\begin{aligned} |u(x,r,\theta) - f(x,e^{i\theta})| &= |\textstyle\int_0^{2\pi} P(r,\theta-t)f(x,e^{it})dt - \int_0^{2\pi} P(r,\theta-t)f(x,e^{i\theta})dt| \\ &\leq \textstyle\int_0^{2\pi} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt. \end{aligned}$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,e^{it}) - f(x,e^{i\theta})| < \epsilon \text{ for } |t-\theta| < \delta \text{ and all } x,$$

since f is uniformly continuous on the compact space $X \times S^1$. Let I_δ denote the integral

$$\int_0^{2\pi} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt \text{ over the interval } |t-\theta| \leq \delta,$$

and let I'_δ denote this integral over the complement of the interval $|t-\theta| \leq \delta$ in an interval of length 2π. Then we have

$$I_\delta \leq \int_{|t-\theta|\leq\delta} P(r,\theta-t)\epsilon dt \leq \epsilon \int_0^{2\pi} P(r,\theta-t)dt = \epsilon.$$

By (a) the maximum value of $P(r,\theta-t)$ on $|t-\theta| \geq \delta$ is $P(r,\delta)$. Hence

$$I'_\delta \leq P(r,\delta) \int_0^{2\pi} |f(x,e^{it}) - f(x,e^{i\theta})|dt.$$

The integral here as a uniform bound for all x and θ since f is bounded. Thus by (c) we can make

$$I'_\delta \leq \epsilon \text{ by taking } r \text{ close enough to } 1.$$

Therefore

$$|u(x,r,\theta) - f(x,\theta)| \leq I_\delta + I'_\delta \leq 2\epsilon.$$

□

Now we are ready for the proof of the proposition.

Proof of Proposition 23.8. Choosing a Hermitian inner product on E, the endomorphisms of $E \times S^1$ form a vector space $\mathrm{End}(E \times S^1)$ with a norm

$$\|\alpha\| = \sup_{|v|=1} |\alpha(v)|.$$

Note that the triangle inequality holds for the sup-norm, so balls in $\mathrm{End}(E \times S^1)$ are convex. The subspace $\mathrm{Aut}(E \times S^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map

$$\mathrm{End}(E \times S^1) \to [0,\infty),\ \alpha \mapsto \inf_{(x,z) \in X \times S^1} |\det(\alpha(x,z))|.$$

Hence in order to prove the first statement of the proposition it will suffice to show that the Laurent polynomials are dense in $\mathrm{End}(E \times S^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy

$$t\ell + (1-t)f \text{ through clutching functions}$$

which is in $\mathrm{Aut}(E \times S^1)$ for all $0 \leq t \leq 1$. Hence f is homotopic to ℓ in $\mathrm{Aut}(E \times S^1)$ and

$$[E,f] \cong [E,\ell].$$

The second statement follows similarly by approximating a homotopy from ℓ_0 to ℓ_1, viewed as an automorphism of $E \times S^1 \times I$ by a Laurent polynomial homotopy ℓ'_t. Then we can combine these approximations with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1.

Hence we need to show that every $f \in \mathrm{End}(E \times S^1)$ can be approximated by Laurent polynomial endomorphisms. Therefor we choose open sets U_i covering X together with isomorphisms

$$h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}.$$

We may assume that h_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors.

Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$ an let $\{X_i\}$ be the support of ϕ_i. Since X is compact, we cann choose $\{\phi_i\}$ such that each X_i is a compact subset in U_i. Via h_i, the linear maps $f(x,z)$ for $x \in X_i$ can be viewed as matrices. The entries of these matrices define functions

$$X_i \times S^1 \to \mathbb{C}.$$

Applying Lemma 23.9 to each entry of the matrices, we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x \in X_i$. It follows that ℓ_i approximates f in the $\|\cdot\|$-norm, since the entries are uniformly approximated. From the Laurent polynomial approximations ℓ_i over X_i we form the convex linear combination

$$\ell = \sum_i \phi_i \ell_i,$$

which is a Laurent polynomial approximating f over all of $X \times S^1$. □

24. Proof of the Periodicity Theorem II

We continue the sketch of the proof of the periodicity theorem for complex K-theory.

Theorem 24.1. *The natural homomorphism*

$$\mu\colon K(X)\otimes \mathbb{Z}[H]/(H-1)^2 \to K(X)\otimes K(S^2) \to K(X\times S^2)$$

is an isomorphism of rings.

The proof on a careful analysis of the construction of complex vector bundles on $X \times S^2$ via clutching functions. We conclude the proof today with an outline of the ideas.

24.1. **Laurent polynomial clutching functions.** The first step is to reduce to *Laurent polynomial* clutching functions, which have the form

$$\ell(x,z) = \sum_{|i|\le n} a_i(x) z^i$$

where $a_i \colon E \to E$ is a map which restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. Such an a_i will be called an *endomorphism* of E.

Note: The linear transformation $a_i(x)$ is not required to be invertible, hence the terminology. Nevertheless, the linear combination $\sum_{|i|\le n} a_i(x)z^i$ must be invertible, since clutching functions are automorphisms.

Hence the first step is to prove the following simplification.

Proposition 24.2. *Every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy*

$$\ell_t(x,z) \sum_{|i|\le n} a_i(x,t)z^i.$$

The proof is based on the fact that on a compact space X, we can approximate continuous functions $f\colon X\times S^1 \to \mathbb{C}$ by Laurent polynomial functions of the form

$$\sum_{|n|\le N} a_n(x)z^n = \sum_{|n|\le N} a_n(x)e^{in\theta},$$

where $z = e^{i\theta} \in S^1$ and each a_n is a continuous function $X \to \mathbb{C}$. Motivated by Fourier series, we set

$$a_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,e^{i\theta})e^{-in\theta}d\theta.$$

For positive real r, consider the series

$$u(x,r,\theta) = \sum_{n\in\mathbb{Z}} a_n(x)r^{|n|}e^{in\theta}.$$

For fixed $r < 1$, this series converges absolutely and uniformly as (x,θ) ranges over $X \times [0,2\pi]$. This follows from the fact that the geometric series

$$\sum_n r^n$$

converges, and, since $X \times S^1$ is compact,

$$|f(x, e^{i\theta})| \text{ is bounded and hence also } |a_n(x)|.$$

Now we need to show that $u(x,r,\theta)$ approaches $f(x,e^{i\theta})$ uniformly in x and θ as r goes to 1. For then sums of finitely many terms in the series for $u(r,x,\theta)$ with r near 1 will give the desired approximations to f by Laurent polynomial functions. The proof of the following lemma can be found in the notes of the previous lecture (and of course in Hatcher's lecture notes).

Lemma 24.3. *As $r \to 1$, $u(r,x,\theta) \to f(x,e^{i\theta})$ uniformly in x and θ.*

Now we are ready for the proof of the proposition.

Proof of Proposition 23.8. Choosing a Hermitian inner product on E, the endomorphisms of $E \times S^1$ form a vector space $\mathrm{End}(E \times S^1)$ with a norm

$$\|\alpha\| = \sup_{|v|=1} |\alpha(v)|.$$

Note that the triangle inequality holds for the sup-norm, so balls in $\mathrm{End}(E\times S^1)$ are convex. The subspace $\mathrm{Aut}(E\times S^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map

$$\mathrm{End}(E \times S^1) \to [0,\infty),\ \alpha \mapsto \inf_{(x,z)\in X\times S^1} |\det(\alpha(x,z))|.$$

Hence in order to prove the first statement of the proposition it will suffice to show that the Laurent polynomials are dense in $\mathrm{End}(E \times S^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy

$$t\ell + (1-t)f \text{ through clutching functions}$$

which is in $\mathrm{Aut}(E\times S^1)$ for all $0 \leq t \leq 1$. Hence f is homotopic to ℓ in $\mathrm{Aut}(E\times S^1)$ and

$$[E,f] \cong [E,\ell].$$

The second statement follows similarly by approximating a homotopy from ℓ_0 to ℓ_1, viewed as an automorphism of $E \times S^1 \times I$ by a Laurent polynomial homotopy ℓ'_t. Then we can combine these approximations with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1.

Hence we need to show that every $f \in \mathrm{End}(E \times S^1)$ can be approximated by Laurent polynomial endomorphisms. Therefor we choose open sets U_i covering X together with isomorphisms

$$h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{C}^{n_i}.$$

We may assume that h_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors.

Let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$ an let $\{X_i\}$ be the support of ϕ_i, which is a compact subset in U_i. Via h_i, the linear maps $f(x,z)$ for $x \in X_i$ can be viewed as matrices. The entries of these matrices define functions $X_i \times S^1 \to \mathbb{C}$. Applying Lemma 24.3 to each entry of the matrices, we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x \in X_i$. It follows that ℓ_i approximates f in the $\|\cdot\|$-norm, since the entries are uniformly approximated. From the Laurent polynomial approximations ℓ_i over X_i we form the convex linear combination

$$\ell = \sum_i \phi_i \ell_i,$$

which is a Laurent polynomial approximating f over all of $X \times S^1$. □

Now we are reduced to Laurent polynomial clutching functions. In fact, we are reduced to polynomial clutching functions, since if ℓ is a Laurent polynomial we can write it as

$$\ell = z^{-m} q \text{ for a polynomial function q and some } m.$$

Then we get

$$[E,\ell] \cong [E,q] \otimes \hat{H}^{-m}.$$

The next step is to simplify from polynomials to linear clutching functions.

Proposition 24.4. *If q is a polynomial clutching function of degree at most n, then*

$$[E,q] \oplus [nE,\mathrm{id}] \cong [(n+1)E, L^n q] \text{ for a linear clutching function } L^n q.$$

Proof. Let

$$q(x,z) = a_n(x)z^n + \cdots + a_0(x).$$

Each of the matrices

$$A = \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix}$$

defines an endomorphism of $(n+1)E$ by interpreting the (i,j)-entry of the matrix as a linear map from the jth summand of $(n+1)E$ to the ith summand, with the entries 1 denoting the identity $E \to E$ and z denoting z times the identity, for $z \in S^1$.

Now we define the sequence $q_r(z) = q_r(x,z)$ inductively by

$$q_0 = q,\ zq_{r+1}(z) = q_r(z) - q_r(0).$$

Then we have the following matrix identity:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & 1 & 0 & \cdots & 0 & 0 \\ q_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ q_{n-1} & 0 & 0 & \cdots & 1 & 0 \\ q_n & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix} \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

We can rewrite this identity as

$$A = (1 + N_1)B(1 + N_2) \tag{8}$$

where N_1 and N_2 are nilpotent. If N is nilpotent, then $1 + tN$ is an invertible matrix for $0 \le t \le 1$. Since matrix B defines a clutching function for

$$[E,q] \oplus [nE, \mathrm{id}],$$

it is invertible in each fiber. Hence (8) shows that A is invertible in each fiber. Thus A defines an automorphism of $(n+1)E$ for each $z \in S^1$ and therefore a clutching function which we denote by $L^n q$. Since $L^n q$ has the form

$$L^n q(x,z) = a(x)z + b(x),$$

Moreover, it follows from (8) that A and B define homotopic clutching functions. Hence we obtain an isomorphism of vector bundles:

$$[E,q] \oplus [nE, \mathrm{id}] \cong [(n+1)E, L^n q].$$

□

24.2. **Linear clutching functions.** For linear clutching functions we have the following key fact:

Proposition 24.5. *Let* $a, b \in \mathrm{End}(E)$ *and assume we are given a bundle* $[E, a(x)z+b(x)]$*. Then there is a splitting* $E \cong E_- \oplus E_+$ *with*

$$[E, a(x)z + b(x)] \cong [E_+, z] \oplus [E_-, \mathrm{id}] (\cong E_+ \otimes H \oplus E_-).$$

To prepare the proof of the proposition, we start with a brief side discussion. Let T be an endomorphism of a finite dimensional vector space E, and let S be a circle in the complex plane which does not pass through any eigenvalue of T. Then

$$Q = \frac{1}{2\pi i} \int_S (z - T)^{-1} dz$$

is a projection operator in E, i.e., $Q^2 = Q$, which commutes with T. This induces a decomposition

$$E = E_+ \oplus E_-, \; E_+ = QE \text{ and } E_- = (1 - Q)E,$$

which is invariant under T. Hence T can be written as

$$T = T_+ \oplus T_-.$$

Moreover, the eigenvalues of T_+ are all inside S, while the eigenvalues of T_- are all outside of S.

Sketch of a proof of Proposition 24.5. For a,$b \in \mathrm{End}(E)$, write $p(x) = a(x)z + b(x)$. Since $a(x)z + b(x)$ is invertible for all x, $b(x)$ has no eigenvalues on the unit circle S^1. We define an endomorphism of E by

$$Q = \frac{1}{2\pi i} \int_{|z|=1} (az + b)^{-1} a dz.$$

(Hence Q defines a linear transformation on each fiber E_x of E.) It is even a projection operator. Moreover, Q commutes with a and b. Now one defines

$$E_+ = QE \text{ and } E_- = (1 - Q)E.$$

Now one has to check that E_+ and E_- inherit a vector bundle structure from E. Once this is done, we get a decomposition

$$E \cong E_+ \oplus E_-$$

and our endomorphisms induce endomorphisms

$$p_+ = a_+z + b_+ \in \mathrm{End}(E_+ \times S^1) \text{ and } p_- = a_-z + b_- \in \mathrm{End}(E_- \times S^1).$$

Moreover, a_+ and b_- are isomorphisms (and so are a_- and b_+.) Setting

$$p^t = p^t_+ + p^t_-, \text{ where } p^t_+ = a_+z + tb_+, \; p^t_- = ta_-z + b_-, \; 0 \le t \le 1,$$

we obtain isomorphisms

$$\begin{aligned} [E,p] &\cong [E, a_+z + b_-] \text{ from the homotopies above} \\ &\cong [E_+, a_+z] \oplus [E_-, b_-] \\ &\cong [E_+, z] \oplus [E_-, \mathrm{id}] \text{ since } a_+z \sim z \text{ and } b_- \sim \mathrm{id} \end{aligned}$$

□

24.3. **Proof of the Periodicity Theorem.** As a consequence of the previous discussion we obtain that for every vector bundle F over $X \times S^2$ there is an integer $n \geq 0$ and bundles E_1, E_2 and E_3 over X such that

$$F \otimes H^n \oplus \pi^* E_1 \cong \pi^* E_2 \otimes H \oplus \pi^* E_3,$$

where $\pi\colon X \times S^2 \to X$ is the projection.

Moreover, the homotopy of clutching functions

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

implies

$$H^2 \oplus 1 = H \oplus H.$$

Hence we have

$$([H] - 1)^2 = ([H]^{-1} - 1)^2 = 0 \text{ in } K(X \times S^2).$$

Finally, this implies that every element ξ in $K(X \times S^2)$ can be written as

$$\xi = \pi^* \xi_1 + \pi^* \xi_2^2 \cdot ([H] - 1)$$

with $\xi_1, \xi_2 \in K(X)$. This shows the surjectivity statement of the Periodicity Theorem.

The injectivity can then be proved by showing that the elements ξ_1 and ξ_2 are in fact unique in $K(X)$. One has to check that all the choices we made during the constructions did not matter. We omit the careful analysis that is necessary to do this. We refer to Atiyah's book orHatcher's lecture notes for more details.

In the end, the Periodicity Theorem tells us that $K(X \times S^2)$ is a free $K(X)$-module with generators 1 and $[H] - 1$. The ring structure on $K(X \times S^2)$ is determined by the single relation $([H] - 1)^2 = 0$.

25. Adams operations in complex K-theory

There are very important ring homomorphisms in complex K-theory, called *Adams operations.* Today we are going to see how they can be defined and that they have the following properties:

Theorem 25.1. *For each non-zero integer k and each compact Hausdorff space X, there is a ring homomorphism*

$$\psi^k \colon K(X) \to K(X)$$

satisfying the following properties:

(1) $\psi^1 = \mathrm{id}$ *and* ψ^{-1} *is induced by conjugation of complex bundles.*
(2) $\psi^k f^* = f^* \psi^k$ *for all maps* $f \colon X \to Y$*, i.e., the* ψ^k *are natural homomorphisms.*
(3) $\psi^k(L) = L^k = L \otimes \cdots \otimes L$ *if* L *is a line bundle.*
(4) $\psi^k \circ \psi^\ell = \psi^{k\ell}$.
(5) $\psi^p(\alpha) \equiv \alpha^p$ *modulo* p *for a prime* p
(6) *If* X *is a based space, then, by the naturality property (2), each* ψ^k *restricts to an operation*

$$\psi^k \colon \tilde{K}(X) \to \tilde{K}(X),$$

since $\tilde{K}(X)$ *is the kernel of the homomorphism* $K(X) \to K(x_0)$.
For $2n$*-spheres, the Adams operations act as*

$$\psi^k(x) = k^n x \text{ for } x \in \tilde{K}(S^{2n}).$$

The proof of the theorem will occupy the rest of today's lecture.

First of all, if we impose property (4), $\psi^{-k} = \psi^k \psi^{-1}$, and use (1) to define ψ^{-1}, we only need to construct the ψ^k for $k > 1$.

By extending the construction from vector spaces to bundles we can form an exterior power $\lambda^k(E)$ which has the following properties:

(i) $\lambda^k(E_1 \oplus E_2) \cong \oplus_{i+j=k} \lambda^i(E_1) \otimes \lambda^j(E_2)$.
(ii) $\lambda^0(E) = 1$, the trivial line bundle.
(ii) $\lambda^1(E) = E$.
(iv) $\lambda^k(E) = 0$ for k greater than the maximum dimension of the fibers of E.

Lemma 25.2. *The* λ^k *extend to operations on* K*-theory*

$$\lambda^k \colon K(X) \to K(X).$$

Proof. Consider the multiplicative group G of power series with constant term 1 in the ring $K(X)[[t]]$ of formal power series in the variable t. We define a function from equivalence classes of vector bundles to this abelian group by setting

$$\Lambda(E) := 1 + \lambda^1(E)t + \cdots + \lambda^k(E)t^k + \cdots .$$

Property (i) above implies

$$\Lambda(E_1 \oplus E_2) = \Lambda(E_1)\Lambda(E_2).$$

This means that Λ is a morphism of monoids and hence induces a homomorphism of groups

$$\Lambda\colon K(X) \to G.$$

We define

$$\lambda^k(x) \text{ to be the coefficient of } t^k \text{ in } \Lambda(x).$$

□

Back to the Adams operations. Let us consider the special case of a vector bundle E which is a sum of line bundles L_i. Then properties (3) and (4) give us a formula

$$\psi^k(L_1 + \cdots + L_n) = L_1^k + \cdots + L_n^k.$$

The construction of the ψ^k will be based on showing that there is a polynomial Q_k with integral coefficients with

$$L_1^k + \cdots + L_n^k = Q_k(\lambda^1(E), \ldots, \lambda^k(E)).$$

This leads us to define

$$\psi^k(E) = Q_k(\lambda^1(E), \ldots, \lambda^k(E))$$

for arbitrary E.

So we need to find these polynomials Q_k. Therefor we consider the polynomial algebra $\mathbb{Z}[x_1, \ldots, x_n]$ and let

$$\sigma_i = x_1 x_2 \cdots x_i + \cdots$$

be the ith elementary symmetric function in the x_i's. The σ_i's form a subring

$$\mathbb{Z}[\sigma_1, \ldots, \sigma_n] \subset \mathbb{Z}[x_1, \ldots, x_n],$$

and satisfy

$$(1 + x_1) \cdots (1 + x_n) = 1 + \sigma_1 + \cdots + \sigma_n.$$

The crucial property for us is that every *symmetric* polynomial of degree k in $x_1, \ldots, x_n$ can be expressed as a unique polynomial in $\sigma_1, \ldots, \sigma_k$. In particular, there is a polynomial Q_k such that

$$Q_k(\sigma_1, \ldots, \sigma_k) = x_1^k + \cdots + x_n^k. \tag{9}$$

Moreover, this Q_k is independent of n as long $k \leq n$, since we can pass from n to $n-1$ by setting $x_n = 0$.

Lemma 25.3. *The Q_k satisfy the recursive formula*

$$Q_k = \sigma_1 Q_{k-1} - \sigma_2 Q_{k-2} + \cdots + (-1)^{k-2}\sigma_{k-1}Q_1 + (-1)^{k-1}k\sigma_k.$$

Proof. This is an exercise. □

The lemma yields for example

$$Q_1 = \sigma_1,\ Q_2 = \sigma_1^2 - 2\sigma_2,\ Q_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

Lemma 25.4. *For $E = L_1 + \cdots + L_n$:*

$$L_1^k + \cdots + L_n^k = Q_k(\lambda^1(E), \ldots, \lambda^k(E)).$$

Proof. The assumption on E implies

$$\Lambda(E) = \prod_i \Lambda(L_i) = \prod_i (1 + \lambda^1(L_i)t) = \prod_i (1 + L_i t).$$

When we compute the product we see that the coefficient $\lambda^i(E)$ of t^i in $\Lambda(E)$ satisfies

$$\lambda^i(E) = \sigma_i(L_1, \ldots, L_n).$$

Substituting L_i for x_i in (9) now yields the assertion. □

Now we can define ψ^k.

Definition 25.5. For every element ξ in $K(X)$ we define

$$\psi^k(\xi) = Q_k(\lambda^1(\xi), \ldots, \lambda^k(\xi)).$$

Now we need to show that the ψ^k's satisfy the properties of the theorem. To do this we will use the following fact, known as the Splitting Principle, which is very useful for proving all kinds of statements in $K(X)$.

Theorem 25.6. *Given a vector bundle $E \to X$ over a compact Hausdorff space X, there is a compact Hausdorff space $F(E)$ and a map $p\colon F(E) \to X$ such that the induced map $p^*\colon K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

Using Theorem 25.6 we finish the proof of Theorem 34.7:

(1) holds by definition for ψ^{-1} and follows from $Q_1 = \sigma_1$ and Theorem 25.6 for ψ^1.
(2) follows from the naturality of λ^k, i.e., $f^*(\lambda^i(E)) = \lambda^i(f^*(E))$.

(3) If $E = L$ is a line bundle, then $\lambda^1(L) = L$ and $\lambda^k(L) = 0$ for $k \geq 2$. Hence

$$\psi^k(L) = Q_k(L) = L^k.$$

For it follows from Lemma 25.3 that $Q_k \equiv \sigma_1^k$ modulo terms in the ideal generated by the σ_i's for $i > 1$.

Additivity: Let E and F be vector bundles over X. By (2) and Theorem 25.6 we take a pullback to split E and then take another pullback to split F as sums of line bundles. But then the identity

$$\psi^k(L_1 + \cdots + L_n) = L_1^k + \cdots L_n^k$$

shows us that ψ^k is additive for sums of line bundles. The injectivity statement of Theorem 25.6 implies that we have

$$\psi^k(E \oplus F) = \psi^k(E) + \psi^k(F).$$

This implies that ψ^k is an additive map $K(X) \to K(X)$.

Multiplicativity: Let E and F be vector bundles over X. By (2) and Theorem 25.6 we take a pullback to split E of line bundles L_i's and then take another pullback to split F as sums of line bundles M_j's. Then $E \otimes F$ is a sum of line bundles $L_i \otimes M_j$. Hence

$$\psi^k(E \otimes F) = \sum_{i,j} \psi^k(L_i \otimes M_j) = \sum_{i,j} (L_i \otimes M_j)^k = \sum_i L_i^k \sum_j M_j^k = \psi^k(E)\psi^k(F).$$

This implies that ψ^k is a multiplicative map $K(X) \to K(X)$.

(4) Theorem 25.6 and Additivity reduce us to the case $E = L$ a line bundle. But in this case we know

$$\psi^k(\psi^\ell(L)) = L^{k\ell} = \psi^{k\ell}(L).$$

(5) Once again we can assume $E = L_1 + \cdots + L_n$. Then

$$\psi^p(E) = L_1^p + \cdots + L_n^p \equiv (L_1 + \cdots + L_n)^p = E^p \text{ modulo } p.$$

(6) We know from before that $\tilde{K}(S^2)$ is generated by $1-[H]$ with $(1-[H])^2 = 0$. By additivity, we know

$$\psi^k(1 - [H]) = 1 - [H]^k.$$

By induction on k, one sees $1 - [H]^k = k(1 - [H])$. For

$$1-[H]^k = (1-[H]^{k-1})[H] + (1-[H]) = (k-1)(1-[H]) + (1-[H]) = k(1-[H]).$$

This shows the formula for S^2. Now we use that

$$S^{2n} = S^2 \wedge \cdots \wedge S^2$$

and $\tilde{K}(S^{2n})$ is generated by the k-fold tensor power

$$(1 - [H]) \otimes \cdots \otimes (1 - [H]).$$

Now (6) follows from the multiplicativity of ψ^k.

26. The Hopf invariant one problem via K-theory

We return to one of our initial problems and answer the question for which n there can be a division algebra structure on $\mathbb{R}^n$. The answer to this question will follow from the solution of a famous problem in algebraic topology, the *Hopf invariant one problem.*

26.1. **The Hopf invariant.** For $n \geq 2$, let S^n be an oriented n-sphere. Assume we are given a pointed map $f\colon S^{2n-1} \to S^n$. Considering S^{2n-1} as the boundary of an oriented $2n$-cell, we can form the cell complex $X = X_f = S^n \cup_f e^{2n}$, the *cofiber of* f. It is the complex formed from the disjoint union of S^n and e^{2n} by identifying each point in $S^{2n-1} = \mathring{e}^{2n}$ with its image under f. The cell complex X has a single vertex, a single n-cell and a single $2n$-cell.

Let

$$\pi\colon X \to X/S^n \cong S^{2n}$$

be the quotient map that collapses S^n. It fits into a sequence

$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{i} X \xrightarrow{\pi} S^{2n} \xrightarrow{\Sigma f} S^{n+1}.$$

Now we specialize to the case that n is *even* and form the long exact sequence in reduced K-theory of the pair (X, S^n). Since

$$\tilde{K}^1(S^{2n}) = \tilde{K}^1(S^n) = 0$$

we obtain a short exact sequence

$$0 \to \tilde{K}(S^{2n}) \xrightarrow{\pi^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(S^n) \to 0. \tag{10}$$

Let i_n be a generator of $\tilde{K}(S^n)$ and i_{2n} be a generator of $\tilde{K}(S^{2n})$. Choose an element

$$a \in \tilde{K}(X) \text{ such that } i^*(a) = i_n \text{ and let } b = \pi^*(i_{2n}) \in \tilde{K}(X).$$

The sequence (12) shows that $\tilde{K}(X)$ is a free abelian with generators a and b, since

$$\tilde{K}(S^{2n}) \cong \tilde{K}(S^n) \cong \mathbb{Z}.$$

Since any square in $\tilde{K}(S^n)$ vanishes we have $i_n^2 = 0$. Hence

$$a^2 = h(f) \cdot b \text{ for some integer } h(f).$$

Lemma 26.1. *The integer $h(f)$ is well-defined.*

Proof. We need to show that $h := h(f)$ does not depend on the choice of a. Because of the exactness of (12), a is unique up to adding a multiple of b. Moreover,

$$(a + mb)^2 = a^2 + 2m \cdot a \cdot b, \text{ since } b^2 = \pi^*(i_{2n}^2) = 0.$$

Hence it suffices to show $a \cdot b = 0$. Since b maps to 0 in $\tilde{K}(S^n)$, so does $a \cdot b$. Hence

$$a \cdot b = k \cdot b \text{ for some integer } k.$$

Multiplying the equation $k \cdot b = b \cdot a$ on the right by a gives

$$k \cdot b \cdot a = b \cdot a^2 = b \cdot h \cdot b = h \cdot b^2 = 0 \text{ since } b^2 = 0.$$

Thus $k \cdot b \cdot a = 0$, which implies $a \cdot b = 0$ since $a \cdot b$ lies in the image of $\tilde{K}(S^{2n})$ in $\tilde{K}(X)$ which is an infinite cyclic subgroup of $\tilde{K}(X)$. □

Definition 26.2. The *Hopf invariant* of f is the integer $h(f)$.

Example 26.3. If n is 2, 4, or 8, there exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant one. For $n = 2$, f may be taken as the natural projection

$$f\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1,$$

viewing S^3 as the unit sphere in the complex plane $\mathbb{C}^2$. Such an f is the attaching map in the complex projective plane

$$\mathbb{C}\mathrm{P}^2 = S^2 \cup_f e^4.$$

Then we have $h(f) = 1$, since $\tilde{K}(\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot a^2$, and hence the generator b is exactly a^2.

The cases $n = 4$ and $n = 8$ correspond to the quaternionic plane and the Cayley plane, respectively. We will get back to these examples later.

Remark 26.4. The Hopf invariant is usually defined using integral cohomology groups. But we will show later that both definitions yield the same number. Using the cohomological definition it is clear that, if n is odd, then $h(f) = 0$ for all f. So n even is the only interesting case and our initial reduction to that case is not really a restriction.

Remark 26.5. The homotopy type of X depends only on the homotopy class of the map f. Thus $h(f)$ only depends on the homotopy class of f. We may therefore speak of the Hopf invariant of a homotopy class and consider h as a function

$$h\colon \pi_{2n-1}(S^n) \to \mathbb{Z}.$$

The Hopf invariant has the following properties.

Proposition 26.6. *Let $n \geq 2$ be an even integer. The Hopf invariant has the following properties:*

(1) *If* $g\colon S^{2n-1}\to S^{2n-1}$ *has degree* d*, then* $h(f\circ g)=d\cdot h(f)$.
(2) *If* $e\colon S^n\to S^n$ *has degree* d*, then* $h(e\circ f)=d^2\cdot h(f)$.
(3) *There exists a map* $f\colon S^{2n-1}\to S^n$ *with Hopf invariant two.*
(4) *The Hopf invariant defines a homomorphism of groups* $\pi_{2n-1}(S^n)\to\mathbb{Z}$.

We will postpone the proof of the proposition. We just mention an immediate consequence for the structure of the homotopy groups of spheres.

Corollary 26.7. *If* n *is even, then* $\pi_{2n-1}(S^n)$ *contains an infinite cyclic subgroup as a direct summand.*

Proof. In fact, the cyclic subgroup generated by the homotopy class of a map of Hopf invariant two must be mapped isomorphically onto the even integers by the homomorphism h. □

The much more important and harder result is the following famous theorem of J. F. Adams. Adams' initial proof was based on cohomological methods. Using Adams operations in complex K-theory yields a much simpler proof due to Adams and Atiyah.

Theorem 26.8. *For an even integer* $n\geq 2$*, there exists a map* $f\colon S^{2n-1}\to S^n$ *with* $h(f)=\pm 1$ *only if* $n=2,\ 4,$ *or* 8.

Proof. We write $n=2m$. Since we computed the effect of the kth Adams operation ψ^k on $\tilde{K}(S^{2m})$ we know

$$\psi^k(i_{2n})=k^{2m}i_{2n} \text{ and } \psi^k(i_n)=k^m i_n.$$

Hence

$$\psi^k(b)=k^{2m}b \text{ and } \psi^k(a)=k^m a+\mu_k$$

for some integer μ_k. For $k=2$ this is

$$2^m a+\mu_2 b=\psi^2(a)\equiv a^2=h(f)\cdot b \bmod 2.$$

Thus $h(f)=\pm 1$ implies that μ_2 is odd.

Now, for any odd k,

$$\begin{aligned}\psi^k\psi^2(a) &= \psi^k(2^m a+\mu_2 b)\\ &= k^m 2^m a+(2^m\mu_k+k^{2m}\mu_2)b\end{aligned}$$

while

$$\begin{aligned}\psi^2\psi^k(a) &= \psi^2(k^m a+\mu_k b)\\ &= 2^m k^m a+(k^m\mu_2+2^{2m}\mu_k)b.\end{aligned}$$

Since $\psi^k\psi^2 = \psi^{2k} = \psi^2\psi^k$, these two expressions must be equal. Moreover, since $\tilde{K}(X)$ is a free abelian group, the coefficients of b must agree

$$2^m(2^m - 1)\mu_k = k^m(k^m - 1)\mu_2.$$

Since μ_2 is odd, this implies that 2^m divides $k^m - 1$. Already with $k = 3$, the following elementary number theoretic lemma shows that this implies $m = 1$, 2, or 4. □

Lemma 26.9. *If* 2^m *divides* $3^m - 1$ *then* $m = 1$, 2, *or* 4.

Proof. Write $m = 2^\ell k$ with k odd. It suffices to show that the highest power of 2 dividing $3^m - 1$ is 2 for $\ell = 0$ and $2^{\ell+2}$ for $\ell > 0$. Then the lemma follows, since if 2^n divides $3^m - 1$, then we deduce $m \leq \ell + 2$, hence $2^\ell \leq 2^\ell k = m \leq \ell + 2$. This implies $\ell \leq 2$ and $m \leq 4$. The cases $m = 1$, 2, 3, and 4 can then be checked individually.

We use induction on ℓ. For $\ell = 0$ we have

$$3^m - 1 = 3^k - 1 \equiv 2 \mod 4, \text{ since } 3 \equiv -1 \mod 4 \text{ and } k \text{ is odd.}$$

Hence the highest power of 2 dividing $3^m - 1$ is 2. In the next case $\ell = 1$, we have

$$3^m - 1 = 3^{2k} - 1 = (3^k - 1)(3^k + 1).$$

The highest power of 2dividing the first factor is 2 as we just showed and the highest power of 2 dividing the second factor is 2 since

$$3^k + 1 \equiv 4 \mod 8 \text{ because } 3^2 \equiv 1 \mod 8 \text{ and } m \text{ is odd.}$$

So the highest power of 2 dividing the product $(3^k - 1)(3^k + 1)$ is 8. For the inductive step of passing from ℓ to $\ell + 1$ with $\ell \geq 1$, or in other words from m to $2m$ with m even, write

$$3^{2m} - 1 = (3^m - 1)(3^m + 1).$$

Then $3^m + 1 \equiv 2$ mod 4 since m is even, so the highest power dividing $3^m + 1$ is 2. Thus the highest power of 2 dividing $3^{2m} - 1$ is twice the highest power of 2 dividing $3^m - 1$. □

27. Consequences of the Hopf invariant one problem

Last time we discussed the K-theoretical proof of the following fundamental result.

Theorem 27.1. *For an even integer $n \geq 2$, there exists a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant one only if $n = 2$, 4, or 8.*

Today we will see some consequences of this result.

27.1. **H-space structures on S^{n-1}.** As an important consequence of the theorem we can determine for which n the sphere S^n admits an H-space structure, i.e., there is a continuous multiplication map

$$g\colon S^n \times S^n \to S^n$$

with a two-sided identity element.

Theorem 27.2. *If S^{n-1} is an H-space, then $n = 1$, 2, 4, or 8.*

Let us first deal with the case that n is *odd*. Write $n - 1 = 2k$. Since the K-theory group $K(S^{2k})$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^2)$, the Bott periodicity theorem implies

$$K(S^{2k} \times S^{2k}) \cong \mathbb{Z}[\alpha,b]/(\alpha^2,\beta^2)$$

where α and b denote the pullback of generators of $K(S^{2k})$ and $K(S^{2k})$ under the projections of $S^{2k} \times S^{2k}$ onto its two factors. An additive basis for $K(S^{2k} \times S^{2k})$is thus $\{1, \alpha, \beta, \alpha\beta\}$.

Now let us assume we had an H-space multiplication map

$$\mu\colon S^{2k} \times S^{2k} \to S^{2k}$$

and let e be the identity element. The induced homomorphism of K-rings has the form

$$\mu^*\colon \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2).$$

We claim

$$\mu^*(\gamma) = \alpha + \beta + m\alpha\beta \text{ for some integer } m.$$

For: the composition

$$S^{2k} \xrightarrow{i} S^{2k} \times S^{2k} \xrightarrow{\mu} S^{2k}$$

is the identity, where i is the inclusion onto either of the subspaces $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$ (with e the identity element of the H-space structure). The map i^* for i the inclusion onto the first factor sends α to γ and b to 0, so the coefficient of α in $\mu^*(\gamma)$ must be 1. Similarly the coefficient of β in $\mu^*(\gamma)$ must be 1. This proves the claim.

But this leads to a contradiction, since it implies

$$\mu^*(\gamma^2) = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0,$$

which is impossible since $\gamma^2 = 0$.

The strategy to prove Theorem 27.2 for n *even* is the following: given an H-space structure on S^{n-1}, we construct from it a map $f\colon S^{2n-1} \to S^n$ of Hopf invariant one.

Let $g\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous map. Regard S^{2n-1} as

$$\partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n,$$

and we consider S^n as the union of two disks D^n_+ and D^n_- with their boundaries identified. Then $f\colon S^{2n-1} \to S^n$ is defined by

$$f(x,y) = |y| g(x, y/|y|) \in D^n_+ \text{ on } \partial D^n \times D^n$$

and

$$f(x,y) = |x| g(x/|x|, y) \in D^n_- \text{ on } D^n \times \partial D^n.$$

Note that f is well-defined and continuous, even when $|x|$ or $|y|$ is zero, and f agrees with g on $S^{n-1} \times S^{n-1}$.

Lemma 27.3. *Let $n \geq 2$ be an even integer. If $g\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ is an H-space multiplication, then the associated map $f\colon S^{2n-1} \to S^n$ has Hopf invariant ± 1.*

Proof. Let $e \in S^{n-1}$ be the identity element for the H-space multiplication, and let f be the map constructed above. In view of the definition of f it is natural to view the characteristic map ϕ of the $2n$-cell of X_f as a map

$$\phi\colon (D^n \times D^n, \partial(D^n \times D^n)) \to (X_f, S^n).$$

In the following commutative diagram the horizontal maps are the product maps. The diagonal map is the external product, equivalent to the external product

$$\tilde{K}(S^n) \otimes \tilde{K}(S^n) \to \tilde{K}(S^{2n}),$$

which is an isomorphism since it is an iterate of the Bott periodicity isomorphism.

$$
\begin{array}{ccc}
\tilde{K}(X_f)\otimes\tilde{K}(X_f) & \longrightarrow & \tilde{K}(X_f) \\
\cong\uparrow & & \uparrow \\
\tilde{K}(X_f,D^n_-)\otimes\tilde{K}(X_f,D^n_+) & \longrightarrow & \tilde{K}(X_f,S^n) \\
\phi^*\otimes\phi^*\downarrow & & \phi^*\downarrow\cong \\
\tilde{K}(D^n\times D^n,\partial D^n\times D^n)\otimes\tilde{K}(D^n\times D^n,D^n\times\partial D^n) & \longrightarrow & \tilde{K}(D^n\times D^n,\partial(D^n\times D^n)) \\
\downarrow\cong & \nearrow\cong & \\
\tilde{K}(D^n\times\{e\},\partial D^n\times\{e\})\otimes\tilde{K}(\{e\}\times D^n,\{e\}\times\partial D^n) & &
\end{array}
$$

By the definition of an H-space and the definition of f, the map ϕ restricts to a homeomorphism from $D^n\times\{e\}$ onto D^n_+ and from $\{e\}\times D^n$ onto D^n_-. It follows that the element $a\otimes a$ in the upper left group maps to a generator of the group in the bottom row of the diagram, since a maps to a generator of $\tilde{K}(S^n)$ by definition. Therefore by the commutativity of the diagram, the product map in the top row sends

$$a\otimes a\mapsto \pm b$$

since b was defined to be the image of a generator of $\tilde{K}(X_f,S^n)$. Thus we have

$$a^2=\pm b,$$

which means that the Hopf invariant of f is ± 1. □

Theorem 27.2 is now an immediate consequence of the lemma.

27.2. **Division algebra structures on $\mathbb{R}^n$.** The determination of which spheres are H-spaces has the following important implications.

Theorem 27.4. *Let $\omega\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a map with two-sided identity element $e\neq 0$ and no zero-divisors. Then $n=1$, 2, 4, or 8.*

Proof. The product restricts to give $\mathbb{R}^n-\{0\}$ an H-space structure. Since S^{n-1} is homotopy equivalent to $\mathbb{R}^n-\{0\}$, it inherits an H-space structure. Explicitly, we may assume that $e\in S^{n-1}$by rescaling the metric, and we give S^{n-1} the multiplication

$$\phi\colon S^{n-1}\times S^{n-1}\to S^{n-1}$$

defined by

$$\phi(x,y)=\omega(x,y)/|\omega(x,y)|.$$

This is well-defined, since ω has no zero divisors. □

Remark 27.5. Note that ω need not be bilinear, just continuous. it also need not have a strict unit. All we needed is that e is a two-sided unit up to homotopy for the restriction of ω to $\mathbb{R}^n - \{0\}$.

In Lecture 3, we showed that there are trivializations of the tangent bundle of the spheres S^1, S^3, and S^7. Now we can show that there are no other spheres with trivial tangent bundle.

Theorem 27.6. *If S^n is parallelizable, i.e., if the tangent bundle τ to S^n is trivial, then $n = 0$, 1, 3, or 7.*

Proof. The case $n = 0$ is trivial. So let $n \geq 1$ and assume that S^n is parallelizable. Let $v_1, \ldots, v_n$ be a tangent vector field which are linearly independent at each point of S^n. By the Gram-Schmidt process we may make the vectors $x, v_1(x), \ldots, v_n(x)$ orthonormal for all $x \in S^n$. We may assume also that at the first standard basis vector e_1, the vectors $v_1(e_1), \ldots, v_n(e_1)$ are the standard basis vectors $e_2, \ldots, e_{n+1}$. To achieve this we might have to change the sign of v_n to get the orientations right and then deform the vector fields near e_1.

Now let $\phi_x \in SO(n+1)$ send the standard basis to $x, v_1(x), \ldots, v_n(x)$. Then the map

$$\phi\colon (x,y) \mapsto \phi_x(y)$$

defines an H-space structure on S^n with the identity element e_1 since ϕ_{e_1} is the identity map and $\phi_x(e_1) = x$ for all x. Hence $n = 1$, 3, or 7. $\square$

28. The Chern character

We have seen that singular cohomology and K-theory enjoy similar properties. The splitting principle implies a direct connection between them which we will describe in today's lecture.

28.1. **The Chern character.** Let X be a compact Hausdorff space. We want to define a ring homomorphism, called *Chern character*, from K-theory to cohomology.

Before we define this homomorphism we think of an assignment that sends vector bundles to cohomology classes, the Chern classes. We need to understand how the tensor product of line bundles behaves under Chern classes. Recall

$$\mathbb{C}\mathrm{P}^\infty \simeq K(\mathbb{Z},2)$$

and that line bundles are classified by their Chern classes regarded as elements of

$$[X,\mathbb{C}\mathrm{P}^\infty] \cong H^2(X;\mathbb{Z}).$$

The tensor product of two line bundles is represented by a product map

$$\phi\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$$

which gives $\mathbb{C}\mathrm{P}^\infty$ an H-space structure. We may think of ϕ as an element of

$$H^2(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \oplus H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$$

and this element is the sum of the Chern classes in the two copies of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$

This shows that for two line bundles L_1 and L_2 over X, we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Now we would like to define a ring homomorphism $ch\colon K(X) \to H^*(X;\mathbb{Q})$. We start with the case of a line bundle $L \to X$. We want ch to send the tensor product to products in in cohomology. So we set

$$ch(L) = e^{c_1(L)} = 1 + c_1(L) + c_1(L)/2! + \cdots \in H^*(X;\mathbb{Q}),$$

because then

$$ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = ch(L_1) \cdot ch(L_2).$$

(If the sum defining $ch(L)$ has infinitely many terms, it will not lie in the direct sum but rather in the direct product of the groups $H^*(X;\mathbb{Q})$. But in the main examples, $H^n(X;\mathbb{Q})$ will be zero for n sufficiently large.)

For a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$ we define

$$ch(E) = \sum_i ch(L_i) = \sum_i e^{t_i} = n + (t_1 + \cdots + t_n) + \cdots + (t_1^k + \cdots + t_n^k)/k! + \cdots$$

where $t_i = c_1(L_i)$. The total Chern class $c(E)$ is then

$$c(E) = (1 + t_1) \cdots (1 + t_n) = 1 + c_1(E) + \cdots c_n(E)$$

and $c_j(E) = \sigma_j$ is the jth elementary symmetric polynomial in the t_i's.

As we saw in Lecture 25, there is a polynomial Q_k with

$$Q_k(\sigma_1, \ldots, \sigma_k) = t_1^k + \cdots + t_n^k.$$

Hence the above formula reads

$$ch(E) = \dim E + \sum_{k>0} Q_k(c_1(E), \ldots, c_k(E))/k!.$$

For general E, we define $ch(E)$ by this formula.

Remark 28.1. In fact, if we want to define ch as a natural ring homomorphism which sends "generators for spheres to generators" then we have only one chance to do this. For, assume ch is such a map. Then for $X = S^2 = \mathbb{C}\mathrm{P}^1$

$$ch\colon K(S^2) \to H^*(S^2; \mathbb{Q})$$

the generator $H - 1$ is sent to a generator x in $H^2(S^2; \mathbb{Q})$, hence H is sent to $1 + x$ in $H^*(S^2; \mathbb{Q})$. For $\mathbb{C}\mathrm{P}^\infty$ this implies

$$ch\colon K(\mathbb{C}\mathrm{P}^\infty) \to H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q}), H \mapsto 1 + x + \cdots = f(x)$$

where $f(x)$ is some power series in x. Now looking at the commutative diagram

$$\begin{array}{ccc} K(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) & \longrightarrow & K(\mathbb{C}\mathrm{P}^\infty) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Q}) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q}) \end{array}$$

we see that the series f must satisfy $f(x + y) = f(x) \cdot f(y)$, where y is the label for the generator of the cohomology of the other copy of $\mathbb{C}\mathrm{P}^\infty$. But there is only one power series that does the job, namely $f(x) = e^x$.

28.2. **A more formal description of** ch**.** Let R be a any commutative ring and consider a formal power series

$$f(t) = \sum_i a_i t^i \in R[[t]].$$

Given an element $x \in H^n(X; R)$, we let

$$f(x) = \sum a_i x^i \in H^{**}(X; R),$$

where $H^{**}(X;R) = \prod_i H^i(X;R)$ whose elements are considered as formal sums $\sum_i y_i$ with $\deg(y_i) = i$.

Via the splitting principle we can use f to construct a natural homomorphism of abelian monoids

$$\hat{f}\colon \mathrm{Vect}(X) \to H^{**}(X;R)$$

For a line bundle L over X, we set

$$\hat{f}(L) = f(c_1(L)).$$

For a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles over X, we set

$$\hat{f}(E) = \sum_{i=1}^{n} f(c_1(L)).$$

For a general n-plane bundle E over X, we let $\hat{f}(E)$ be the unique element of $H^{**}(X;R)$ such that

$$p^*(\hat{f}(E)) = \hat{f}(p^*(E)) \in H^{**}(F(E);R).$$

More explicitly, writing $p^*E = L_1 \oplus \cdots L_n$, we know by the definition of Chern classes

$$\prod_{1\leq k\leq n} (x - c_1(L_k)) = 0.$$

This implies that

$$c_k(p^*E) = p^*(c_k(E)) = \sigma_k(c_1(L_1), \ldots, c_1(L_n))$$

is the kth elementary symmetric polynomial in the $c_1(L_k)$. Likewise, we see that $\hat{f}(p^*E)$ is a symmetric polynomial in the $c_1(L_i)$ and can therefore be written as a polynomial in the elementary symmetric polynomials. Applying this polynomial to the $c_k(E)$ gives the element $\hat{f}(E) \in H^{**}(X;R)$. For a vector bundle E over a non-connected space X, we add the elements obtained by restricting E to the components of X. By the naturality property of $K(X)$, $\hat{f}$ extends to a homomorphism

$$\hat{f}\colon K(X) \to H^{**}(X;R).$$

There is also an analogous multiplicative extension $\bar{f}$ of f that starts from the definition

$$\bar{f}(E) = \prod_{i=1}^{n} f(c_1(L_i))$$

on a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles.

As an example, we look at the following special case.

Lemma 28.2. *For any R, if $f(t) = 1 + t$, then $\bar{f}(E) = c(E)$ is the total Chern class of E.*

Proof. For a line bundle, we have $\bar{f}(L) = 1 + c_1(L) = c(L)$, and for a sum $E = L_1 \oplus \cdots \oplus L_n$ of line bundles we get

$$\bar{f}(E) = \prod_i (1 + c_1(L_i)) = 1 + c_1(E) + \cdots + c_n(E)$$

since $c_k(E)$ is equal to the kth elementary symmetric function in the $c_1(L_i)$'s. Hence if E is an arbitrary bundle, then

$$\bar{f}(E) = 1 + c_1(E) + \cdots + c_n(E) = c(E).$$

□

The example we are interested in is the Chern character which gives rise to an isomorphism between rationalized K-theory and rational cohomology.

Definition 28.3. For $R = \mathbb{Q}$ and $f(t) = e^t = \sum_i t^i/i!$, we define the *Chern character*

$$ch(E) \in H^{**}(X; \mathbb{Q}) \text{ by } ch(E) = \hat{f}(E).$$

It is clear that both descriptions of ch agree.

28.3. **Properties of ch.** This allows us to prove the following result.

Proposition 28.4. *The Chern character is a ring homomorphism*

$$ch\colon K(X) \to H^{**}(X; \mathbb{Q}).$$

Proof. By the splitting principle and the construction of ch it suffices to check this when E_1 and E_2 are sums of line bundles. In this case we have

$$ch(E_1 \oplus E_2) = ch(\oplus_{i,j} L_{ij}) = \sum e^{c_1(L_{ij})} = ch(E_1) + ch(E_2)$$

and

$$ch(E_1 \otimes E_2) = ch(\oplus_{j,k}(L_{1j} \otimes L_{2k})) = \sum ch(L_{1j} \otimes L_{2k}) = \sum ch(L_{1j}) \cdot ch(L_{2k}) = ch(E_1) \cdot ch(E_2).$$

□

Proposition 28.5. *For $n \geq 1$, the Chern character maps $\tilde{K}(S^{2n})$ isomorphically onto the image of $H^{2n}(S^{2n}; \mathbb{Z})$ in $H^{2n}(S^{2n}; \mathbb{Q})$.*

Proof. Since $ch(x \otimes (H-1)) = ch(x) \cdot ch(h-1)$ we have the commutative diagram

$$\begin{array}{ccc} \tilde{K}(X) & \xrightarrow{\cong} & \tilde{K}(S^2 \wedge X) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \tilde{H}^*(X;\mathbb{Q}) & \xrightarrow{\cong} & \tilde{H}^{*+2}(S^2 \wedge X;\mathbb{Q}) \end{array}$$

where the upper map is the external tensor product with $H-1$, and the lower map is the product with

$$ch(H-1) = ch(H) - ch(1) = 1 + c_1(H) - 1 = c_1(H),$$

which is a generator of $H^2(S^2;\mathbb{Z})$. Hence the lower map is an isomorphism too and even restricts to an isomorphism with $\mathbb{Z}$-coefficients. Taking $X = S^{2n}$, the result follows by induction on n, starting with the trivial case $n = 0$. □

Corollary 28.6. *A class in $H^{2n}(S^{2n};\mathbb{Z})$ occurs as a Chern class $c_n(E)$ if and only if it is divisible by $(n-1)!$.*

Proof. For vector bundles $E \to S^{2n}$ we have $c_1(E) = \cdots = c_{n-1}(E) = 0$, so

$ch(E) = \dim E + Q_n(c_1, \ldots, c_n)/n! = \dim E \pm nc_n(E)/n! = \dim E \pm c_n(E)/(n-1)!$

by the recursive formula for Q_n we mentioned in Lecture 25

$$Q_n = \sigma_1 Q_{n-1} - \sigma_2 Q_{n-2} + \cdots + (-1)^{n-2}\sigma_{n-1}Q_1 + (-1)^{n-1}n\sigma_n.$$

□

Now since Chern classes are in even degrees, the image of ch lies in the sum of the even degree elements in $H^{**}(X;\mathbb{Q})$ which we denote by $H^{even}(X;\mathbb{Q})$. We define $H^{odd}(X;\mathbb{Q})$ to be the sum of the odd degree elements. Then we can extend ch to $\mathbb{Z}/2$-graded reduced cohomology by defining ch on $\tilde{K}^1(X)$ to be the composite

$$\tilde{K}^1(X) \cong \tilde{K}(\Sigma X) \xrightarrow{ch} \tilde{H}^{even}(\Sigma X;\mathbb{Q}) \cong \tilde{H}^{odd}(X;\mathbb{Q}).$$

Then we can prove the following fundamental result.

Theorem 28.7. *For any pointed finite CW-complex X, ch induces an isomorphism*

$$\tilde{K}^*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{H}^{**}(X;\mathbb{Q}).$$

Sketch of the proof. We think of both the source and the target as $\mathbb{Z}/2$-graded. The Proposition 36.5 implies the conclusion when $X = S^n$ for any n. The crucial point is that the map of the theorem is part of a natural transformation of cohomology theories. Then the assertion follows from the result for $X = S^n$, the five lemma and induction on the number of cells of X.

More explicitely, the case of a cell complex with a single cell is trivial. Then if X is obtained from a subcomplex A by attaching a cell, then we get a sequence

$$X/A \to S^1 \wedge A \to S^1 \wedge X \to (S^1 \wedge X)/(S^1 \wedge A) \to S^2 \wedge A.$$

Applying the Chern character to this sequence yields a commutative diagram of five-term exact sequence (tensoring with $\mathbb{Q}$ is exact). Now the spaces X/A and $(S^1 \wedge X)/(S^1 \wedge A)$ are spheres, and both $S^1 \wedge A$ and $S^2 \wedge A$ are both cell complexes with the same number of cells as A (we collapse the suspension or double suspension of a 0-cell). The five-lemma gives us the result for $S^1 \wedge X$. Then we obtain the result for X by replacing X with $S^1 \wedge X$ in the above argument and using that ch commutes with double suspension. □

29. The e-invariant

Today we are going to elaborate a little bit more on the construction we used for the Hopf invariant one problem. It turns out that this picture contains much more information.

29.1. **Getting information about maps between spheres.** Let us look at a slight variation of the way we defined the Hopf invariant using K-theory. For $m, n \geq 1$, let

$$f \colon S^{2n+2m-1} \to S^{2n}$$

be a pointed map. Let

$$X = X_f = S^{2n} \cup_f e^{2n+2m}$$

be the mapping cone of f, $i \colon S^{2n} \hookrightarrow X$ be the inclusion, and

$$\pi \colon X \to X/S^{2n} \cong S^{2n+2m}$$

be the map that collapses S^{2n}. We would like to measure the extend to which f is not null, i.e., not homotopic to a constant map. Therefor we would like to use our favorite (at least for the moment) cohomology theory, complex K-theory.

As in Lecture 26, the sequence

$$S^{2n+2m-1} \xrightarrow{f} S^{2n} \xrightarrow{i} S^{2n} \cup_f e^{2n+2m} \xrightarrow{\pi} S^{2n+2m}$$

(or rather the pair (X, S^{2n})) induces a long exact sequence in reduced K-theory. Since the K-theory of spheres is concentrated in even degrees, the K-theory degree of f, i.e., $\tilde{K}(f)$, is zero. For our goal to measure the extend to which f is not null this is bad news. But there is still some more information to exploit.

Since $\tilde{K}(f) = 0$, we obtain a short exact sequence

$$0 \to \tilde{K}(S^{2n+2m}) \xrightarrow{\pi^*} \tilde{K}(S^{2n} \cup_f e^{2n+2m}) \xrightarrow{i^*} \tilde{K}(S^{2n}) \to 0. \quad (11)$$

We know that the outermost groups are the integers and the group in the middle is an extension. We would like to understand how far from the trivial extension the sequence (12). In order to make this more precise we need to think a little bit more about what kind of groups we are talking about.

We have already noticed that the outermost groups in (12) are the integers. But we also know that the Adams operation ψ^k acts on $\tilde{K}(S^{2n})$ by k^n and it acts on $\tilde{K}(S^{2n+2m})$ by k^{n+m}. So let us write $\mathbb{Z}(n)$ for the first group and $\mathbb{Z}(n+m)$ for the second. We want to consider them in some category of "abelian groups with Adams operations".

Let us make an informal definition:

Definition 29.1. An *abelian group with Adams operations* is an abelian group A together with morphisms $\psi^k\colon A \to A$, for $k \in \mathbb{Z}$, which commute with each other and satisfy $\psi^\ell \psi^k = \psi^{k\ell}$.

But we can say even a little bit more about the K-theory groups. In the previous lecture we defined the Chern character

$$ch\colon K(Y) \to \oplus_n H^{2n}(Y;\mathbb{Q})$$

which becomes an isomorphism after tensoring $K(Y)$ with $\mathbb{Q}$ (assuming Y is a finite cell complex). The splitting principle now tells us that the Adams operations on cohomology are given by

$$\psi^k = k^n \text{ on } H^{2n}(Y;\mathbb{Q}).$$

To check this, write a bundle E as a sum of line bundles. Then we only need to compute the effect of ψ^k on the $2n$th component ch^n of $ch(L)$ for a line bundle. Then we have $\psi^k(L) = L^k$, and hence

$$ch^n(\psi^k(L)) = ch^n(L^k) = (c_1(L^k))^n/n! = (kc_1(L))^n/n! = k^n c_1(L)^n/n! = k^n ch^n(L).$$

Hence the action of the Adams operations is *semisimple* on rational K-theory. In other words, if A is in the image of the K-theory functor, then $A \otimes \mathbb{Q}$ is a big sum of copies of $\mathbb{Q}(n)$.

29.2. **The e-invariant as an extension.** Now let us get back to the geometric situation. The short exact sequence (12) corresponds to an element $e(f)$ ("e" for *extension*) in

$$\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$$

where the Ext is in the category of abelian groups together with Adams operations.

What can we say about this group $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n+m))$? The short exact sequence

$$0 \to \mathbb{Z}(n+m) \to \mathbb{Q}(n+m) \to \mathbb{Q}/\mathbb{Z}(n+m) \to 0$$

induces a long exact sequence of Ext-groups

$$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n{+}m)) \to \mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n{+}m)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Z}(n{+}m)) \to \mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n{+}m)).$$

Lemma 29.2. *For $m \neq 0$, the two outermost groups* $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+m))$ *and* $\mathrm{Ext}^1(\mathbb{Z}(n), \mathbb{Q}(n+m))$ *are zero.*

Proof. We only prove the first assertion. If there is a non-trivial homomorphism $\mathbb{Z}(n) \to \mathbb{Q}(n+m)$, then $1 \in \mathbb{Z}(n)$ is sent to some element $\alpha \in \mathbb{Q}(n+m)$, and thus k^n would have to be sent to $k^{n+m}\alpha$ which is a contradiction. Hence

$\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}(n+m)) = \{0\}$. The second assertion requires a little bit more work. Since the discussion is more philosophical for the moment, we skip the proof. □

As a consequence of the lemma we get an isomorphism

$$\mathrm{Hom}(\mathbb{Z}(n),\mathbb{Q}/\mathbb{Z}(n+m)) \cong \mathrm{Ext}^1(\mathbb{Z}(n),\mathbb{Z}(n+m)).$$

The group $\mathrm{Hom}(\mathbb{Z}(n),\mathbb{Q}/\mathbb{Z}(n+m))$ is a subgroup of $\mathbb{Q}/\mathbb{Z}$ and consists of things compatible with the Adams operations.

In order to understand this group a bit more, let us spell out what we know. A homomorphism

$$\mathbb{Z}(n) \to \mathbb{Q}/\mathbb{Z}(n+m)$$

is determined by where it sends $1 \in \mathbb{Z}(n)$. Let us call the image $x \in \mathbb{Q}/\mathbb{Z}(n+m)$. Then x has to satisfy a condition in order to make the map a homomorphism of abelian groups with Adams operations. Namely, for all k, we must have

$$(k^{n+m} - k^n) \cdot x = 0 \in \mathbb{Q}/\mathbb{Z},$$

because this expresses the compatibility with ψ^k. This means that the denominator of x must divide all the numbers $(k^{n+m} - k^n)$ for all k.

In other words, the group $\mathrm{Ext}^1(\mathbb{Z}(n),\mathbb{Z}(n+m))$ is cyclic of order

$$\text{the greatest common divisor of } k^n(k^m - 1) \text{ for all } k.$$

Hence we should calculate this greatest common divisor. There is a nice answer for it. But before we do this let us make things a bit more concrete. We should also think about the specific element in $\mathrm{Ext}^1(\mathbb{Z}(n),\mathbb{Z}(n+m))$ that sequence (12) produces.

29.3. **The e-invariant as an element in $\mathbb{Q}/\mathbb{Z}$.** Let i_{2n} be a generator of $\tilde{K}(S^{2n})$ and i_{2n+2m} be a generator of $\tilde{K}(S^{2n+2m})$. Choose an element

$a \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$ such that $i^*(a) = i_{2n}$ and let $b = \pi^*(i_{2n+2m}) \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$.

Then for any k, we have

$$\psi^k(a) = k^n \cdot a + \mu_k \cdot b.$$

Since the Adams operations commute, we must have

$$\psi^k(\psi^\ell(a)) = \psi^k(\ell^n a + \mu_\ell b) = \ell^n k^n a + \ell^n \mu_k b + k^{n+m} \mu_\ell b = \ell^n k^n a + k^n \mu_\ell b + \ell^{n+m} \mu_k b = \psi^\ell(\psi^k(a))$$

and hence

$$k^n(k^m - 1)\mu_\ell = \ell^n(\ell^m - 1)\mu_k$$

for any k and ℓ. This shows us that the rational number

$$e(f) := \frac{\mu_k}{k^n(k^m-1)} \in \mathbb{Q}.$$

is independent of k. But it might depend on our choice of a. If we change a by a multiple of b, then $e(f)$ is changed by an integer. (For $a' = a + p \cdot b$, we get $e'(f) = e(f) + p$.) Thus $e(f)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$.

Finally, recalling where we started we see that we have produced an assignment

$$(f\colon S^{2n+2m-1} \to S^{2n}) \mapsto e(f) \in \mathbb{Q}/\mathbb{Z}.$$

Remark 29.3. 1. The map e is called the *e-invariant.* It plays an important role in understanding the structure of the (stable) homotopy groups of the sphere. To get further into this story we introduce in the next lecture the J-homomorphism.
2. That $e(f)$ is an element in $\mathbb{Q}/\mathbb{Z}$ fits well with our discussion above. To determine an element in $\mathrm{Hom}(\mathbb{Z}(n), \mathbb{Q}/\mathbb{Z}(n+m))$ we needed to determine the image of 1 in $\mathbb{Q}/\mathbb{Z}(n+m)$.

Lemma 29.4. *If $f \sim g$, then $e(g) = e(f)$, i.e., e induces a map*

$$e\colon \pi_{2n+2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}.$$

Proof. This follows from applying the functor $\tilde{K}$ to the diagram

$$\begin{array}{ccccccccc}
S^{2n+2m-1} & \xrightarrow{f} & S^{2n} & \xrightarrow{i} & S^{2n} \cup_f e^{2n+2m} & \xrightarrow{\pi} & S^{2n+2m} & \xrightarrow{\Sigma(f)} & S^{2n+1} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
S^{2n+2m-1} & \xrightarrow{g} & S^{2n} & \xrightarrow{i'} & S^{2n} \cup_g e^{2n+2m} & \xrightarrow{\pi'} & S^{2n+2m} & \xrightarrow{\Sigma(g)} & S^{2n+1}.
\end{array}$$

□

30. The e-invariant and the J-homomorphism

We are trying to detect interesting maps between spheres. Last time we defined the e-invariant and showed that we should think of it as an element in some Ext group of abelian groups with Adams operations. This group is finite and cyclic and we saw a criterion for determining its order. But we still need to determine this order. The reason why this is so interesting is that the order will tell us something about the size of some of the stable homotopy groups of spheres.

Let us recall the setup. For $m, n \geq 1$, let

$$f \colon S^{2n+2m-1} \to S^{2n}$$

be a pointed map,

$$X = X_f = S^{2n} \cup_f e^{2n+2m}$$

be the mapping cone of f, $i \colon S^{2n} \hookrightarrow X$ be the inclusion, and

$$\pi \colon X \to X/S^{2n} \cong S^{2n+2m}$$

the map that collapses S^{2n}. This gives us a short exact sequence

$$0 \to \tilde{K}(S^{2n+2m}) \xrightarrow{\pi^*} \tilde{K}(S^{2n} \cup_f e^{2n+2m}) \xrightarrow{i^*} \tilde{K}(S^{2n}) \to 0. \tag{12}$$

Let i_{2n} be a generator of $\tilde{K}(S^{2n})$ and i_{2n+2m} be a generator of $\tilde{K}(S^{2n+2m})$. Choose an element

$a \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$ such that $i^*(a) = i_{2n}$ and let $b = \pi^*(i_{2n+2m}) \in \tilde{K}(S^{2n} \cup_f e^{2n+2m})$.

Then for any k, we have

$$\psi^k(a) = k^n \cdot a + \mu_k \cdot b.$$

Since the Adams operations commute, we must have

$$k^n(k^m - 1)\mu_\ell = \ell^n(\ell^m - 1)\mu_k$$

for any k and ℓ. This shows us that the rational number

$$e(f) := \frac{\mu_k}{k^n(k^m - 1)} \in \mathbb{Q}.$$

is independent of k. But it might depend on our choice of a. If we change a by a multiple of b, then $e(f)$ is changed by an integer. Thus $e(f)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$.

The e-invariant defines a map

$$e \colon \pi_{2n+2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}.$$

An alternative description of the e-invariant can be given using the Chern character. The Chern character gives us a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(S^{2n+2m}) & \xrightarrow{\pi^*} & \tilde{K}(X_f) & \xrightarrow{i^*} & \tilde{K}(S^{2n}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\
0 & \longrightarrow & \tilde{H}^*(S^{2n+2m};\mathbb{Q}) & \xrightarrow[\pi^*]{} & \tilde{H}^*(X_f;\mathbb{Q}) & \xrightarrow[i^*]{} & \tilde{H}^*(S^{2n};\mathbb{Q}) & \longrightarrow & 0
\end{array} \tag{13}
$$

whose rows are exact.

Let $y = \pi^*(ch(i_{2n+2m})) \in \tilde{H}^{2n+2m}(X_f;\mathbb{Q})$ and x be an element in $\tilde{H}^{2n}(X_f;\mathbb{Q})$ that maps to the generator $ch(i_{2n})$. Then we have $ch(b) = y$. Let $r(f) \in \mathbb{Q}$ be such that

$$ch(a) = x + r(f) \cdot y \in \tilde{H}^{2n}(X_f;\mathbb{Q}) \oplus \tilde{H}^{2n+2m}(X_f;\mathbb{Q}).$$

Lemma 30.1. *$r(f) = e(f) \in \mathbb{Q}/\mathbb{Z}$.*

Proof. We calculate

$$ch(\psi^k(a)) = ch(k^n \cdot a + \mu_k \cdot b) = k^n \cdot ch(a) + \mu_k \cdot ch(b) = k^n \cdot x + (k^n \cdot r(f) + \mu_k) \cdot y.$$

On the other hand, we have seen above that ψ^k acts on $\tilde{H}^{2n}$ by multiplication by k^n. Hence

$$\psi^k(ch(a)) = k^n ch^n(a) + k^{n+m} ch^{n+m}(a) = k^n \cdot x + k^{n+m} \cdot r(f) \cdot y.$$

Comparing the coefficients of y in both formulas gives

$$\mu_k = r(f) \cdot (k^n(k^m - 1)).$$

□

Lemma 30.2. *The map e is a group homomorphism.*

Proof. Let $X_{f,g}$ be obtained from S^{2n} by attaching two $2n+2m$-cells by f and g, so $X_{f,g}$ contains both X_f and X_g. There is a quotient map

$$Q\colon X_{f+g} \to X_{f,g}$$

collapsing a sphere $S^{2n+2m-1}$ that separates the $2n+2m$-cell of $X_{f,g}$ into a pair of $2n+2m$-cells. (This is also called the “pinching map”.) It induces a commutative diagram

$$
\begin{array}{ccc}
\tilde{K}(X_{f,g}) & \xrightarrow{Q^*} & \tilde{K}(X_{f+g}) \\
{\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\
\tilde{H}^*(X_{f,g};\mathbb{Q}) & \xrightarrow{Q^*} & \tilde{H}^*(X_{f+g};\mathbb{Q}).
\end{array}
$$

In the upper row, the generators b_f and b_g are mapped to b_{f+g} and $a_{f,g}$ is mapped to a_{f+g}. Similarly, in the lower row, the generators y_f and y_g are mapped to y_{f+g} and $x_{f,g}$ is mapped to x_{f+g}. Using the previous lemma it now suffices to work with r and to look at

$$ch(a_{f,g}) = x_{f,g} + r(f)y_f + r(g)y_g$$

and hence

$$ch(a_{f+g}) = x_{f+g} + (r(f) + r(g))y_{f+g}.$$

□

Remark 30.3. The e-invariant is in fact a stable invariant. We know that the mapping cone satisfies $X_{S^2\wedge f} = S^2 \wedge X_f$ and we noticed in the proof of Proposition 28.5 of Lecture 28 that ch commutes with double suspension. This shows that we have a commutative diagram

$$\begin{array}{ccc} \pi_{2n+2m-1}(S^{2n}) & \xrightarrow{S^2\wedge -} & \pi_{2n+2+2m-1}(S^{2n+2}) \\ & {\scriptstyle e}\searrow \quad \swarrow{\scriptstyle e} & \\ & \mathbb{Q}/\mathbb{Z} & \end{array}$$

Hence we can view e also as a homomorphism

$$e\colon \pi^s_{2m-1}(S^0) \to \mathbb{Q}/\mathbb{Z}$$

from the $(2m-1)$-stable homotopy group of the sphere spectrum.

Now we should start to calculate the e-invariant. The maps for which we get the most important results are in the image of the J-homomorphism.

30.1. **The J-homomorphism.** The J-homomorphism is a natural way to construct maps between spheres. Let us first look at the idea of the construction.

Let $O(n)$ be the group of orthogonal $n \times n$-matrices. It acts on the Euclidean n-space $\mathbb{R}^n$ by linear isometries. A linear isometry of $\mathbb{R}^n$ extends to the one-point compactification S^n. Hence there is a natural map

$$J\colon O(n) \to \mathrm{LinIso}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Map}_*(S^n, S^n) = \Omega^n S^n$$

where $\mathrm{Map}_*(-,-)$ denotes the space of basepoint preserving continuous maps (with the compact-open topology). This induces a homomorphism

$$J\colon \pi_k(O(n)) \to \pi_k(\Omega^n S^n) = \pi_{k+n}(S^n).$$

Remark 30.4. There is a little subtlety concerning the above construction of J. For the basepoint of $\Omega^n S^n$ is the constant map at the basepoint. The space $\Omega^n S^n$ has many path components, one for each degree. The image of $O(n)$ lies in the path components $\Omega^n_1 S^n$ and $\Omega^n_{-1} S^n$ of paths of degree ± 1 (remembering that $O(n)$ has two components). The basepoint of $O(n)$, the identity map, goes to the identity map of S^n. Hence the map $O(n) \to \Omega^n S^n$, as described above, is not basepoint preserving. So we should modify the map by "subtracting off" (in some group model for $\Omega^n S^n$) the identity map. Hence we should use

$$J\colon O(n) \to \Omega^n_1 S^n \xrightarrow{-1} \Omega^n_0 S^n.$$

Here is a more concrete way to define the J-homomorphism. Let $k \geq 1$. An element $[f] \in \pi_k(O(n))$ is represented by a family of isometries

$$f_x \in O(n),\ x \in S^k \text{ with } f_x = \text{id when } x \text{ is the basepoint of } S^k.$$

Writing

$$S^{n+k} = \partial(D^{k+1} \times D^n) = S^k \times D^n \cup D^{k+1} \times S^{n-1} \text{ and } S^n = D^n/\partial D^n,$$

let

$$Jf(x,y) = f_x(y) \text{ for } (x,y) \in S^k \times D^n \text{ and } Jf(D^{k+1} \times S^{n-1}) = \partial D^n,$$

where we think of the latter ∂D^n as the basepoint of $D^n/\partial D^n$.

It is easy to see that if $f \simeq g$ then $Jf \simeq Jg$. Hence we obtain a map

$$J\colon \pi_k(O(n)) \to \pi_{k+n}(S^n).$$

Lemma 30.5. *J is a homomorphism.*

Proof. Exercise. □

It is easy to check that if we embed $O(n)$ into $O(n+1)$ this corresponds to taking suspension. Since both groups $\pi_k(O(n))$ and $\pi_{k+n}(S^n)$ are independent of n for $n-1 > k$, we can pass to the limit in n and get the stable J-homomorphism

$$J\colon \pi_k(O) \to \pi^s_k(S^0) = \pi_k(S^0).$$

The image of the J-homomorphism in $\pi_k(S^0)$ is the main part of the stable homotopy groups which is accessible to direct computations.

30.2. **The complex J-homomorphism.** In our computations we will focus on the following complex version of J. We can compose J with the map

$$\pi_k(U) \to \pi_k(O) \text{ induced by the natural inclusions } U(n) \subset O(2n).$$

This defines the stable complex J-homomorphism

$$J_{\mathbb{C}} \colon \pi_k(U) \to \pi_k(S^0).$$

On the groups $\pi_k(S^0)$ we have defined the e-invariant. Our goal now is to compute the e-invariant on the image of $J_{\mathbb{C}}$, i.e., we want to compute the composition

$$e \circ J_{\mathbb{C}} \colon \pi_k(U) \to \mathbb{Q}/\mathbb{Z}.$$

There is the following great result.

Theorem 30.6. *Let $f \colon S^{2k-1} \to U(n)$ represent a generator in $\pi_{2k-1}(U)$. Then*

$$e(J_{\mathbb{C}} f) = \pm \beta_k / k$$

where β_k is defined by the power series

$$\frac{x}{e^x - 1} = \sum_k \frac{\beta_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of β_k/k (that is the denominator when we take β_k/k in reduced form).

31. The image of the J-homomorphism

The stable J-homomorphism $J\colon \pi_k(O) \to \pi_k(S^0)$ is an important tool to produce interesting maps between spheres. Last time we also considered its complex analogue

$$J_{\mathbb{C}}\colon \pi_k(U) \to \pi_k(O) \to \pi_k(S^0)$$

which is a little bit easier to handle. Today we start to prove the following great result:

Theorem 31.1. *If $f\colon S^{2k} \to BU$ represents a generator x_{2k} in $\pi_{2k}(BU)$, then*

$$e(J_{\mathbb{C}} f) = \pm B_k/k$$

where B_k is the kth Bernoulli number defined by the power series

$$\frac{x}{e^x - 1} = \sum_k \frac{B_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of B_k/k (that is the denominator when we take B_k/k in reduced form).

Before we start let us think a bit more about the maps in question. We can rewrite $J_{\mathbb{C}}$ as

$$\pi_{m-1}U = \pi_m BU \cong \tilde{K}^0(S^m) \to \pi_{m-1}(S^0)$$

and it factors through the real J-homomorphism

$$\pi_{m-1}O = \pi_m BO \cong \tilde{KO}^0(S^m) \to \pi_{m-1}(S^0)$$

where $\tilde{KO}^0(S^m)$ denotes the real K-theory of S^m.

The groups $\pi_{m-1}U$ alternate between being $\mathbb{Z}$ and 0: if m is even, then we get $\mathbb{Z}$; if m is odd, then we get 0:

m	1	2	3	4	5	6	7	8
$\pi_{m-1}U$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$.

The homotopy groups $\pi_{m-1}O$ of O show an 8-fold periodicity:

m	1	2	3	4	5	6	7	8
$\pi_{m-1}O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$.

The map

$$\mathbb{Z} \cong \pi_{m-1}U \to \pi_{m-1}O \cong \mathbb{Z}$$

is an isomorphism when $m \equiv 4 \bmod 8$ and is multiplication by 2 when $m \equiv 0 \bmod 8$. (One can see this by looking at the composite $\pi_m BU \to \pi_m BO \to \pi_m BU$.)

We formulated our theorem in terms of the complex J-homomorphism, because it makes things easier. But from the table of $\pi_{m-1}BO$ we see immediately that the J-homomorphism is zero when m is odd.

Moreover, the cost of working with complex rather than real K-theory is an overall factor of two, i.e., by computing

$$\pi_{2k-1}U \xrightarrow{J_{\mathbb{C}}} \pi_{2k-1}S^0 \xrightarrow{e_{\mathbb{C}}} \mathbb{Q}/\mathbb{Z}$$

we get twice the value of the real e-invariant $e_{\mathbb{R}}$ of the real J-homomorphism of the generator of real K-theory.

The theorem tells us that if $x_{2n} \in \pi_{2n}BU$ is a generator, then $e_{\mathbb{C}}(J_{\mathbb{C}}(x_{2n})) = \frac{B_n}{n}$. Then one can deduce from the above discussion the following result.

Corollary 31.2. *If $y_{4n} \in \pi_{4n}BO$ is a generator, then $e_{\mathbb{R}}(J_{\mathbb{R}}(y_{4n})) = \frac{B_{2n}}{4n}$.*

31.1. **Thom complexes and the J-homomorphism.** The initiating idea for the proof of the theorem is based on the following very important fact. If we want to show that a map of spheres is nontrivial, we have to make computations in the mapping cone. When a map is in the image of J, we have a lot of information about this mapping cone: it is actually a Thom complex.

Proposition 31.3. *Let ξ be an n-dimensional complex vector bundle over S^{2k} classified by a map*

$$\xi \colon S^{2k} \to BU(n).$$

The Thom complex of ξ is $S^{2n} \cup_{J\xi} e^{2n+k}$.

Proof. Since $\pi_{2k}(BU(n)) \cong \pi_{2k-1}(U(n))$, there is a map

$$f \colon S^{2k-1} \to U(n).$$

We consider f as a clutching function for ξ. In fact, we can identify ξ with the bundle ξ_f obtained from $D^{2k} \times \mathbb{C}^n \amalg \mathbb{C}^n$ by identifying

$$(x,v) \sim f_x(v) \text{ for } x \in \partial D^{2k}.$$

Restricting to the unit disk bundle $D(\xi_f)$ we have $D(\xi_f)$ expressed as a quotient of $D^{2k} \times D^{2n} \amalg D^{2n}$ by the same relation. The quotient $T(\xi_f) = D(\xi_f)/S(\xi_f)$ contains a sphere $S^{2n} = D^{2n}/\partial D^{2n}$, coming from the second copy of D^{2n}, and $T(\xi_f)$ is obtained from S^{2n} by attaching a cell e^{2k+2n} with characteristic map the quotient map

$$D^{2k} \times D^{2n} \to D(\xi_f) \to T(\xi_f).$$

The attaching map of the cell is precisely $J(f)$, since it is given by

$$(x,v) \mapsto f_x(v) \in D^{2n}/\partial D^{2n} \text{ on } \partial D^{2k} \times D^{2n}$$

and maps all of $D^{2k} \times \partial D^{2n}$ to the point $\partial D^{2n}/\partial D^{2n}$. ∎

If we want to compute $eJ_{\mathbb{C}}(f)$ we need to compute $ch(a)$ for an element

$$a \in \tilde{K}(X_{Jf}) = \tilde{K}(T_\xi) \text{ which restricts to a generator in } \tilde{K}(S^{2n})$$

where S^{2n} is a fiber of $D(\xi)$ as in the previous proof. A class in $\tilde{K}(T(\xi))$ which restricts to a generator for each sphere S^n coming from a fiber of ξ is called a *Thom class* of ξ. Hence we need to understand the Chern character of Thom classes in K-theory.

32. The image of the J-homomorphism and Thom classes

We are still on the way to prove the following theorem on the complex J-homomorphism

$$J_{\mathbb{C}}\colon \pi_k(U) \to \pi_k(O) \to \pi_k(S^0).$$

Theorem 32.1. *If x_{2k} in $\pi_{2k}(BU)$ is a generator, then*

$$e(J_{\mathbb{C}}f) = \pm B_k/k$$

where B_k is the kth Bernoulli number defined by the power series

$$\frac{x}{e^x-1} = \sum_k \frac{B_k x^k}{k!}.$$

Hence the image of J in $\pi_{2k-1}(S^0)$ has order divisible by the denominator of B_k/k (that is the denominator when we take B_k/k in reduced form).

32.1. **Thom classes and the Thom isomorphism in K-theory.** We saw last time that if E is an n-dimensional complex vector bundle over S^{2n} classified by a map

$$f\colon S^{2k} \to BU$$

then the Thom complex of ξ is $S^{2n} \cup_{Jf} e^{2n+k}$.

Hence if we want to compute $eJ_{\mathbb{C}}(f)$ we need to compute $ch(a)$ for an element

$$a \in \tilde{K}(X_{Jf}) = \tilde{K}(T_\xi) \text{ which restricts to a generator in } \tilde{K}(S^{2n})$$

where S^{2n} is a fiber of $D(\xi)$ as in the previous proof. A class in $\tilde{K}(T(\xi))$ which restricts to a generator for each sphere S^n coming from a fiber of ξ is called a *Thom class* of ξ. Hence we need to understand the Chern character of Thom classes in K-theory.

We have seen Thom classes before. But let us briefly recall the basic theory. Let E be a complex vector bundle of dimension n over the compact Hausdorff space X. Let $X^E := T(E) = D(E)/S(E)$ denote the Thom space of E over X. The *Thom class* is an element

$$U \in \tilde{K}^0(X^E)$$

which restricts to a generator under the restriction map

$$\tilde{K}^0(X^E) \to \tilde{K}^0((X^E)_x) \cong \tilde{K}^0(E_x^+) \cong \mathbb{Z}$$

for every $x \in X$, where E_x^+ denotes the one-point compactification of the fiber E_x (it's a $2n$-sphere whence the last isomorphism). There are several natural ways to get such a Thom class. One construction uses the projective bundle formula.

First we remark that we can identify X^E with $\mathbb{P}(E\oplus 1)/\mathbb{P}(E)$. Let V be the vector space given by the fiber E_x over some $x\in X$. Given a line ℓ through the origin in $V\oplus 1$ which does not lie in V, there is a unique point v in V such that $(v,1)\in V\oplus 1$. This defines a map $\mathbb{P}(V\oplus 1)\to V$. The lines that are in V correspond to the point at ∞ in the fiber of the Thom complex of V. Hence we have checked on each fiber that we have an isomorphism

$$X^E=\mathbb{P}(E\oplus 1)/\mathbb{P}(E).$$

Now it is easier to produce the Thom class on the right hand side, because we know that we have the tautological line bundle L over the projective space.

Let L be the canonical line bundle over $\mathbb{P}(E\oplus 1)$. We know that $K^*(\mathbb{P}(E\oplus 1))$ is the free $K^*(X)$-module with basis $1,L,\dots,L^n$. Restricting to $\mathbb{P}(E)\subset\mathbb{P}(E\oplus 1)$, we see that $K^*(\mathbb{P}(E))$ is the free $K^*(X)$-module with basis (the restrictions to $\mathbb{P}(E)$ of) $1,L,\dots,L^{n-1}$. So we have a short exact sequence

$$0\to\tilde{K}^*(X^E)\to K^*(\mathbb{P}(E\oplus 1))\xrightarrow{\rho}K^*(\mathbb{P}(E))\to 0.$$

The map ρ sends L^n to L^n. But in $K^*(\mathbb{P}(E))$ we have the relation

$$\sum_i(-1)^i\lambda^i(E)L^{n-i}=0$$

where the $\lambda^i(E)$ are the Chern classes of E in $K^*(X)$ by definition. The class $U_K\in\tilde{K}^0(X^E)$ that maps to the nonzero element

$$\sum_i(-1)^i\lambda^i(E)L^{n-i}\in K^0(\mathbb{P}(E\oplus 1))$$

is the *Thom class* of E that we were looking for.

Moreover, we get that multiplication by U_K gives the *Thom isomorphism*

$$U_K\colon K^0(X)\cong\tilde{K}^0(X^E)$$

and $\tilde{K}^0(X^E)$ is a free $K^0(X)$-module with one generator U_K.

Remark 32.2. We will also sometimes identify

$$U_K \text{ with } \sum_i(-1)^i\lambda^i(E)L^{n-i} \text{ in } \tilde{K}^0(\mathbb{P}(E\oplus 1)).$$

Note that all this makes sense for virtual bundles too, since it is an isomorphism of modules over $K^0(X)$.

Remark 32.3. The previous discussion applies to any cohomology theory with a projective bundle formula for complex vector bundles. In particular, it applies to $\tilde{H}^{even}(-;\mathbb{Q})$. If $x = x(E) \in H^2(\mathbb{P}(E \oplus 1);\mathbb{Q})$ is an element that restricts to a generator of $H^2(\mathbb{CP}^{n-1};\mathbb{Q})$ in each fiber, then there is the relation

$$\sum_i (-1)^i c_i(E) x^{n-i} = 0 \text{ in } H^*(\mathbb{P}(E);\mathbb{Q}).$$

Hence the element $\sum_i (-1)^i c_i(E) x^{n-i} \in H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$ comes from an element $U_H \in H^{2n}(X^E;\mathbb{Q})$ (where we use that $x(E \oplus 1)$ restricts to $x(E)$). This is the Thom class in cohomology. In $H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$ we can identify U_H with $\sum_i (-1)^i c_i(E) x^{n-i}$. Then we get $U_H \cdot x = 0$ in $H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$, because we know $c_i(E \oplus 1) = c_i(E)$ and hence

$$0 = \sum_i (-1)^i c_i(E \oplus 1) x^{n+1-i} = \sum_i (-1)^i c_i(E) x^{n+1-i} = U_H \cdot x.$$

To prove the theorem we need to calculate $ch(U_K)$. By the splitting principle we may assume that $E = L_1 \oplus \cdots \oplus L_n$ splits as a sum of line bundles. The Thom class $U_H = \sum_i (-1)^i c_i(E) x^{n-i}$ in $\mathbb{P}(E \oplus 1)$ then factors as the product

$$U_H = \prod_i (x - x_i) \in H^*(\mathbb{P}(E \oplus 1);\mathbb{Q})$$

where $x_i = c_1(L_i)$. Similarly, the Thom class in K-theory becomes

$$U_K = \prod_i (L - L_i) \in \tilde{K}^0(\mathbb{P}(E \oplus 1)).$$

Therefore we have

$$ch(U_K) = \prod_i ch(L - L_i) = \prod_i (e^x - e^{x_i}) = U_H \cdot \prod_i (\frac{e^{x_i} - e^x}{x_i - x}).$$

Since $U_H \cdot x = 0$, we can set $x = 0$ and simplify this expression to

$$ch(U_K) = U_H \cdot \prod_i (\frac{e^{x_i} - 1}{x_i}).$$

Since the Thom isomorphism $\vartheta \colon H^*(X;\mathbb{Q}) \to H^*(X^E;\mathbb{Q})$ is given by multiplication with U_H, we get the formula

$$\vartheta^{-1} ch(U_K) = \prod_i (\frac{e^{x_i} - 1}{x_i}) \in H^*(X;\mathbb{Q}).$$

Dealing with such power series becomes easier when we take the logarithm. There is a power series expansion for $\log(\frac{e^y - 1}{y})$ of the form $\sum_k c_k \frac{y^k}{k!}$ for some coefficients c_k since the function $\frac{e^y-1}{y}$ is nonzero at 0. Then we can have

$$\log \vartheta^{-1} ch(U_K) = \log(\prod_i (\frac{e^{x_i}-1}{x_i})) = \sum_i \log(\frac{e^{x_i}-1}{x_i}) = \sum_{i,k} c_k \frac{x_i^k}{k!} = \sum_k c_k ch^k(E)$$

where $ch^k(E)$ is the component of $ch(E)$ in dimension $2k$. The last equation uses the fact that E is the sum of line bundles and the definition of the Chern character for line bundles. The splitting principle then tells us that the formula also holds for arbitrary E.

We need to calculate the coefficients c_k. Therefor we differentiate both sides of

$$\sum_k c_k y^k/k! = \log(\frac{e^y-1}{y}) = \log(e^y-1) - \log y.$$

This yields

$$\begin{aligned} \textstyle\sum_k c_k y^{k-1}/(k-1)! &= \tfrac{e^y}{e^y-1} - y^{-1} \\ &= 1 + \tfrac{1}{e^y-1} - y^{-1} \\ &= 1 - y^{-1} + \textstyle\sum_{k\geq 0} B_k y^{k-1}/k! \\ &= 1 + \textstyle\sum_{k\geq 1} B_k y^{k-1}/k! \end{aligned}$$

where the last equation follows from the fact that $B_0 = 1$. Thus we obtain

$$c_k = B_k/k \text{ for } k > 1 \text{ and } 1 + B_1 = c_1.$$

Since $B_1 = -1/2$, we get $c_1 = 1/2$ and $c_1 = -B_1/1$.

32.2. **The proof of Theorem 32.1.** Now we apply the discussion to the n-dimensional bundle $E \to S^{2k}$ corresponding to the element $x_{2k} \in \pi_{2k}BU$. We choose $U_K \in \tilde{K}^0(X_{Jf}) = \tilde{K}^0((S^{2k})^E)$ as the element mapping to a generator in $\tilde{K}^0(S^{2k})$ (changing signs if necessary). We know

$$ch(U_K) = a + r \cdot b \in H^*(X_{Jf}; \mathbb{Q})$$

and hence

$$\vartheta^{-1} ch(U_K) = 1 + r \cdot s$$

where s is a generator of $H^{2k}(S^{2k}; \mathbb{Q})$ and $r = e(J_{\mathbb{C}} f)$ in $\mathbb{Q}/\mathbb{Z}$. Hence

$$\log \vartheta^{-1} ch(U_K) = r \cdot s$$

since $\log(1+z) = z - z^2/2 + \cdots$ and $s^2 = 0$. On the other hand, we have

$$\log \vartheta^{-1} ch(U_K) = c_k ch^k(E)$$

since $H^{2j}(S^{2k}; \mathbb{Q}) = 0$ for $j \neq k$. Moreover, we showed in Lecture 28 that

$$ch^k(E) = s \in H^{2k}(S^{2k}; \mathbb{Q}).$$

Thus, by comparing the two formulas for $\log \vartheta^{-1} ch(U_K)$ we get

$$e(J_{\mathbb{C}} f) = r = c_k = \pm B_k/k.$$

This finishes the prof of Theorem 32.1.

33. Clifford algebras and vector fields on spheres

This was a guest lecture by Mike Hopkins. Here are my notes of his lecture:

Clifford Algebras and vector fields on spheres

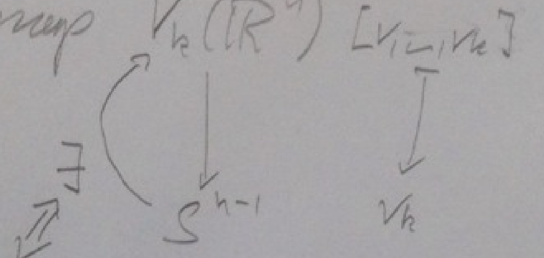
Problem: Determine the maximum number of linearly independent vector fields on S^{n-1}.

Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of k-frames in $\mathbb{R}^n$.

$$V_k(\mathbb{R}^n) = \{[v_1, \dots, v_k] \mid v_i v_j = \delta_{ij}\}$$

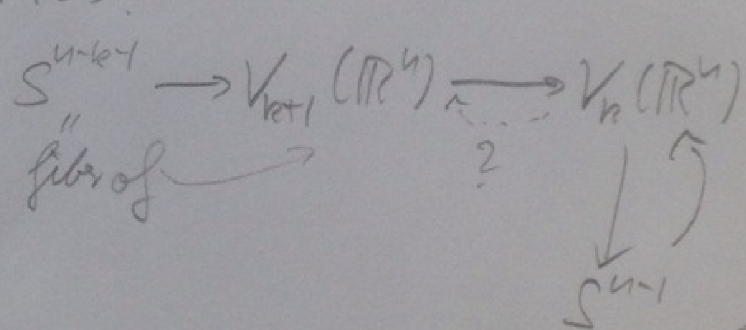
$$= O(n)/O(n-k)$$

Have a map



S^{n-1} has $(k-1)$ linearly independent vector fields.

Can we lift this?:



<u>Obstruction</u> to going further is an elt of $\pi_{n-2} S^{n-k-1}$.

This is the setup.

Let us look at examples:

- We know even spheres have no vector fields ("hairy ball theorem")

- $S^{2n-1} \subset \mathbb{C}^n$

  $v \mapsto iv$ gives a vector field

$S^{4n-1} \subset \mathbb{H}^n$

  $v \mapsto iv, jv, kv$ give 3 v. fields

$S^{8n-1} \subset \mathbb{O}^n \rightsquigarrow 7$ vector fields

This led to the expectation that S^{15} has 15 vector fields...

This is not true. So let us see why:

First a construction:

$$\begin{array}{c} V_{k+1}(\mathbb{R}^n) \\ \downarrow \uparrow \\ S^{n-1} \end{array} \quad \rightsquigarrow k \text{ vector fields} \qquad S^{n-1} \times V \hookrightarrow TS^{n-1} \quad \dim V = k$$

get a map $S^{n-1} \times V \xrightarrow{\varphi} S^{n-1}$ st $\varphi(x, v) \perp x$

or a map $\mathbb{R}^n \times V \longrightarrow \mathbb{R}^n$, think of it as $T_v(x) = \varphi(x, v)$ as a transformation $\mathbb{R}^n \to \mathbb{R}^n \ \forall v$.

- some simplifying assumptions on this map:

  1) bilinear

  2) $T_v^2 = -1$ if $|v| = 1$, or $T_v^2 = -|v|^2$.

<u>Definition</u>: V vector space with $\langle , \rangle$.

$$Cl(V) = \text{free associative algebra on } V / _{v^2 = -|v|^2}$$

"Clifford algebra"

<u>E.g.</u>: $Cl_k = Cl(\mathbb{R}^k, \langle , \rangle)$

$Cl_k' = Cl(\mathbb{R}^k, -\langle , \rangle)$.

- Cl_k = free assoc. algebra on $e_1, \dots, e_k$ modulo
  - $e_i^2 = -1$ — for $i \neq j$: $(e_i + e_j)^2 = -|e_i + e_j|^2 = -2$
  - $e_i e_j = -e_j e_i$ — hence $e_i^2 + e_j^2 + e_i e_j + e_j e_i \Rightarrow e_i e_j = -e_j e_i = 0$

have $\dim Cl_k = 2^k$

- Cl'_k = gen. by $e'_1, \dots, e'_k$ mod $e_i'^2 = +1$, $e'_i e'_j = -e'_j e'_i$

Fact: If Cl_k acts on $\mathbb{R}^n$ then S^{n-1} has k vector fields

	Cl_k	Cl'_k
$k=0$	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R} \leftrightarrow \frac{e_1 - 1}{\sqrt{2}}, \frac{e_1 + 1}{\sqrt{2}}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	
6	$\mathbb{R}(8)$	
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	
8	$\mathbb{R}(16)$	
9		

Notes:

- where $A(n)$ = {$n \times n$-matrices over A} $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Think of $Cl(V)$ as a $\mathbb{Z}/2$-graded algebra
  $v \in V$ has degree 1, $v \cdot v = -|v|^2$

then $Cl(V \oplus W) \cong Cl(V) \hat{\otimes} Cl(W)$ = $\mathbb{Z}/2$-graded tensor product!

Prop: $\cdot\ Cl_k \otimes Cl'_2 = Cl'_{k+2}$

$\cdot\ Cl'_k \otimes Cl_2 = Cl_{k+2}$

This tells us how to complete the table!

Proof of prop: Have generators $e_1, \dots, e_k, e'_1, e'_2$ with $e_i^2 = -1$, $e_i e_j = -e_j e_i$, $e_1'^2 = e_2'^2 = 1$, $\cdot\ e'_1 e'_2 = -e'_2 e'_1$

now need to figure out how they "commute":

e.g. $e_1 e'_1 e'_2 e_1 e'_1 e'_2 = e_1^2 e'_1 e'_2 e'_1 e'_2$

$= -e_1^2 e_1'^2 e_2'^2$

$= 1$.

$Cl'_{k+2} \to Cl_k \otimes Cl'_2$ sends

$e_1 \mapsto e_1 e'_1 e'_2$

$e_k \mapsto e_k e'_1 e'_2$

$e_{k+1} \mapsto e'_1$

$e_{k+2} \mapsto e'_2$ □

Some algebra facts: $\cdot\ \mathbb{H}\otimes\mathbb{C}=\mathbb{C}(2)$ (follows from the theory of central simple algebras)
(that we need)
$\cdot\ \mathbb{H}\otimes\mathbb{H}=\mathbb{R}(4)$

We read off from the table and the prop.:

$\cdot\ Cl_k\otimes Cl_2\otimes Cl_2' = Cl_k\otimes\mathbb{H}(2)$

$\cdot\ Cl_k\otimes\mathbb{R}(16)=Cl_{k+8}$ "Periodicity of Clifford algebras".

k	Cl_k	dim. of smallest real representation	
0	$\mathbb{R}$	1	
1	$\mathbb{C}$	2	
2	$\mathbb{H}$	4	
3	$\mathbb{H}\oplus\mathbb{H}$	4	$\leftarrow S^{4k-1}$ has 3 vector fields
4	$\mathbb{H}(2)$	8,	
5	$\mathbb{C}(4)$	8	
6	$\mathbb{R}(8)$	8	
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	8	$\leftarrow S^{8k-1}$ has 7 vector fields
8	$\mathbb{R}(16)$	16	and 8 vector fields on S^{15}
9	$\mathbb{C}(16)$	32	

This method produces a formula:

write $n=m\cdot 2^{r}$, $(m,2)=1$, $r=4c+d$, $\rho(n)=2^d+8c$:

then there are $\rho(n)-1$ vector fields on S^{n-1}.

$\rho(n)$ is called the <u>nth Radon Hurwitz number</u>.

Let us look at S^{15}:

$$\begin{array}{ccccc} S^6 & \longrightarrow & V_{10}(\mathbb{R}^{16}) & \longrightarrow & V_9(\mathbb{R}^{16}) \\ & & & & \downarrow \uparrow \\ & & & & S^{15} \end{array}$$

The obstruction is in $\pi_{14} S^6 \cong \pi_8^{st}(S^0)$

This obstruction is $J_{\mathbb{R}}(\text{generator of } \pi_9 BO = \mathbb{Z}/2)$.

Adams then showed that this obstruction is nonzero and thereby determined the number of vector fields on S^{15} (and on all spheres).

34. The image of J and the Adams conjecture

34.1. **The image of J.** The stable real J-homomorphism is a map

$$\pi_{k-1}O \to \pi^s_{k-1}(S^0) = \pi_{k-1}S^0.$$

We are interested in the case $k = 4n$ because in those degrees the homotopy groups of O provide the most interesting image in the stable homotopy groups. We saw in the previous lectures that if $x_{4n-1}O$ is a generator then

$$e(Jx_{4n}) = \pm B_{2n}/4n$$

where B_i is the ith Bernoulli number. Hence the order of the image of J in $\pi_{4n-1}S^0$ is divisible by the denominator of $B_{2n}/4n$. Today we want to explore the information of the J-homomorphism a bit further.

Let us denote the denominator of $B_{2n}/4n$ by $m(2n)$. We have a lower bound for the image of J, for the order of $\operatorname{Im} J$ is divisible by $m(2n)$. So what about an upper bound? Adams showed that there is actually an upper bound and thereby determined the image of J in $\pi_{4n-1}S^0$ completely. (Well, almost completely since he could not figure out a possible factor of 2 for $4n \equiv 0 \bmod 8$.) We want to follows Adams' great ideas and see how close he got to determine the image of J.

Adams proved the following result.

Theorem 34.1. *The image $J(\pi_{4n-1}O)$ of the stable J-homomorphism in $\pi_{4n-1}S^0$ is cyclic of order*

(i) *$m(2n)$ if $4n \equiv 4$ modulo 8*
(ii) *$m(2n)$ or $2m(2n)$ if $4n \equiv 0$ modulo 8.*

Remark 34.2. Mahowald showed later that the factor 2 in (ii) is not there. Adams could not settle this factor since he could prove his conjecture only for the complex K-theory and not for the real K-theory of S^{4n}. Adams' conjecture was then proven independently and in full generality by Quillen-Friedlander, Quillen, Sullivan and Becker-Gottlieb. We are going to sketch a proof in the next lecture.

Before we think about a proof, let us first note a consequence of Theorem 34.1. Let $j\colon \operatorname{Im} J \hookrightarrow \pi_{4n-1}S^0$ denote the inclusion. Adams shows that the image of e in $\mathbb{Q}/\mathbb{Z}$ is precisly the subgroup of cosets $z/m(2n)$, $z \in \mathbb{Z}$. Hence we have a commutative diagram

$$\begin{array}{ccccc} & & \pi_{4n-1}S^0 & & \\ & \overset{j}{\nearrow} & & \overset{e}{\searrow} & \\ \operatorname{Im} J & & \xrightarrow{\quad e\circ j \quad} & & \mathbb{Z}/m(2n). \end{array}$$

By Theorem 34.1 and its improvement we know that Im J is cyclic of order $m(2n)$. Therefore the diagram provides a direct sum splitting

$$\pi_{4n-1}S^0 \cong \mathrm{Im}\, J \oplus \mathrm{Ker}\; e.$$

Example 34.3. For $r = 4n - 1$ let us take the generator in $\pi_r SO$ and let its image under $J\colon \pi_r SO \to \pi_r S^0$ be j_r. Then we have:

$$e(j_3) = 1/24,\ e(j_7) = -1/240,\ e(j_{11}) = 1/504,\ e(j_{15}) = -1/480,\ e(j_{19}) = 1/264.$$

For $r = 3, 7, 11$, we have

$$\pi_3 S^0 \cong \mathbb{Z}/24,\ \pi_7 S^0 \cong \mathbb{Z}/240,\ \pi_{11} S^0 \cong \mathbb{Z}/504.$$

Or in other words, the kernel of e is trivial in these cases. But for $r = 15, 19$, the kernel of e is $\mathbb{Z}/2$.

Remark 34.4. Since the numbers $m(2n)$ are unbounded we see that, even though the stable homotopy groups $\pi_r S^0$ are of finite, arbitrarily large orders can occur.

34.2. **Adams' upper bound for Im J.** We know that Im J is divisible by $m(2n)$. To prove Theorem 34.1 we need an argument in the opposite direction.

Let Y be an abelian group with Adams operations, i.e., an abelian group with endomorphisms ψ^k for every $k \in \mathbb{Z}$. A map between such groups is a homomorphism of abelian groups which is compatible with the operations.

Let e be a function that assigns to each pair $k \in \mathbb{Z}$, $y \in Y$ a non-negative integer $e(k, y)$. Then we define Y_e to be the subgroup of Y generated by the elements

$$k^{e(k,y)}(\psi^k - 1)y.$$

It is clear that if

$$e_1 \geq e_2, \text{ then } Y_{e_1} \subseteq Y_{e_2}.$$

Hence we can define

$$J''(X) := Y/\cap_e Y_e$$

where the intersection runs over all functions e.

Remark 34.5. If Y is finitely generated, it is easy to see that it suffices to let e run over the functions f which are independent of y and get the same quotient group $J''(X)$. For it is clear that

$$\cap_e Y_e \subseteq \cap_f Y_f.$$

For $y \in Y$, let $y_1, \ldots, y_n$ generate y. For any function $e(k, y)$ define the corresponding function $f(k)$ by

$$f(k) := \mathrm{Max}_{1 \leq r \leq n} e(k, y_r).$$

It is clear that we have $Y_f \subseteq Y_e$ and hence

$$\cap_f Y_f \subseteq \cap_e Y_e.$$

Moreover, if Y_1 and Y_2 are finitely generated, then we have

$$(Y_1 \oplus Y_2)_f = (Y_1)_f \oplus (Y_2)_f$$

and hence

$$\cap_f (Y_1 \oplus Y_2)_f = \cap_f (Y_1)_f \oplus \cap_f (Y_2)_f.$$

As a consequence we get

$$J''(Y_1 \oplus Y_2) = J''(Y_1) \oplus J''(Y_2).$$

For $Y = K(X)$ we set $J''_{\mathbb{C}}(X) := J''(K(X))$ and for $Y = KO(X)$ we set $J''(X) := J''(KO(X))$. Let

$$r\colon K(X) \to KO(X)$$

be the canonical map. Since it is compatible with the Adams operations, it induces a map

$$J''_{\mathbb{C}}(X) \to J''(X).$$

Proposition 34.6. *a) Let P be a point. Then*

$$J''(P) = \mathbb{Z}.$$

b) If X is a finite cell complex, then

$$J''(X) = \mathbb{Z} + \tilde{J}''(X) \text{ with } \tilde{J}''(X) = J''(\tilde{KO}(X)).$$

Proof. a) We know $KO(P) = \mathbb{Z}$ and the operations are just given by $(\psi^k - 1)y = 0$ for all k and y.
b) We just need to apply part a) and the second part of the above remark. □

Here is the reason why we are interested in the groups $J''(Y)$ for real K-theory. Adams made the following important conjecture. The formulation of the conjecture and its proof require to give a different interpretation of $J(X)$ in terms of spherical fibrations. Since we will need some time to think about these fibrations in more detail, we postpone this interpretation for a moment. Nevertheless we formulate the conjecture in its general form and think for now of the special case $X = S^m$.

The Adams conjecture 34.7 (The Adams Conjecture)**.** *Let X be a finite cell complex, k an integer, and $y \in KO(X)$. Then there exists a non-negative integer $e = e(k, y)$ such that*

$$k^e(\psi^k - 1)y \in \operatorname{Ker} J.$$

Moreover, these elements (for all k) generate the kernel of J.

The consequence of the conjecture for our discussion is the following.

Proposition 34.8. *Suppose for S^{4n} Conjecture 34.7 holds for all k and y. Then $\tilde{J}''(S^{4n})$ is an upper bound for $\operatorname{Im} J$ in the sense that the surjective map $J\colon KO(S^{4n}) \to \operatorname{Im} J$ factors through an epimorphism $\tilde{J}''(S^{4n}) \to \operatorname{Im} J$.*

Example 34.9. Take X to be the sphere S^{4n}. We claim that the group $\tilde{J}''(S^{4n})$ is cyclic of order $m(2n)$. If $y \in \tilde{KO}(S^{4n})$, we have

$$k^{f(k)}(\psi^k - 1)y = k^{f(k)}(k^{2n} - 1)y$$

since ψ^k acts on the K-theory of S^{4n} by multiplication by k^{2n}. (We proved this only for complex K-theory, but the same argument shows it for real K-theory too.) Thus the subgroup Y_f of $\tilde{KO}(S^{4n}) = \mathbb{Z}$ consists of the multiples of $h(f, 2n)$ where $h(f, 2n)$ is the greatest common divisor of the integers

$$k^{f(k)}(k^{2n} - 1), \text{ for all } k \in \mathbb{Z}.$$

But this number is exactly $m(2n)$. Hence $\tilde{J}''(S^{4n}) = \tilde{KO}(S^{4n})/Y_f = \mathbb{Z}/m(2n)$.

34.3. **A comment on Adams' proof of Theorem 34.1.** Adams proved the assertion for the real K-theory of a sphere S^{2n} under the assumption that the map

$$r\colon \tilde{K}(S^{2n}) \to \tilde{KO}(S^{2n})$$

is an epimorphism.

For $4n \equiv 4$ modulo 8, the map

$$r\colon \tilde{K}(S^{4n}) \to \tilde{KO}(S^{4n})$$

is an epimorphism. Hence by Proposition 35.7 $\tilde{J}''_{\mathbb{R}}(S^{4n})$ is an upper bound for $\operatorname{Im} J$. By Example 34.9 this implies that $\tilde{J}_{\mathbb{R}}(S^{4n})$ divides $m(2n)$.

For $4n \equiv 0$ modulo 8 the proof would be the same except that in this case image of the map

$$r\colon \tilde{K}(S^{4n}) \to \tilde{KO}(S^{4n})$$

consists of the elements divisible by 2. For this case Adams could not prove his conjecture for S^{4n} and hence he could not settle the factor 2. We will investigate this further in the next lecture.

35. Sphere bundles and the Adams conjecture

35.1. **Sphere bundles.** Let X be a connected finite cell complex. We saw that the J-homomorphism could be defined by sending an automorphism of $\mathbb{R}^n$ to the induced automorphism of the one-point compactification. Today we want to generalize this construction and study J as a construction on vector bundles as follows.

Let $E \to X$ be an n-dimensional real vector bundle. By taking the *fiberwise one-point compactification* we get an associated fiber bundle $S(E) \to X$ whose fibers are all n-spheres S^n. We call such a bundle a sphere bundle.

We will say that a map $f\colon S(E) \to S(E')$ of bundles is a *fiber homotopy equivalence* if there is a bundle map $g\colon S(E') \to S(E)$ such that $f \circ g$ and $g \circ f$ are homotopic through bundle maps to the respective identities.

Taking the associated sphere bundle of a vector bundle respects direct sums in the sense that

$$S(E \oplus E') \cong S(E) \wedge_X S(E')$$

where $\wedge_X$ denotes the fiberwise smash product.

Definition 35.1. We denote by $\mathcal{SF}(X)$ the Grothendieck group of pointed sphere bundles over X modulo fiber homotopy equivalence. The group law is given by the fiberwise smash product.

Remark 35.2. A fiber bundle whose fibers who are all of the *homotopy type* of a sphere is called a *pointed spherical fibration*. Hence we could have defined $\mathcal{SF}(X)$ also as the Gorthendieck group of (pointed) spherical fibrations.

Sending a vector bundle to its fiberwise one-point compactification defines a homomorphism

$$KO(X) \to \mathcal{SF}(X).$$

Example 35.3. We want to understand this map for X a sphere. A vector bundle over X is determined by its clutching function. This can be expressed as an isomorphism

$$\tilde{KO}(S^n) \cong \pi_{n-1}O.$$

Similarly, a sphere bundle is determined by a clutching function

$$f\colon S^{n-1} \to \mathrm{Homeo}(S^k, S^k).$$

Since we are only interested in sphere bundles modulo fiber homotopy equivalence, it suffices to specify the clutching function up to homotopy equivalence. Hence a function

$$f\colon S^{n-1} \to \mathrm{Equiv}(S^k, S^k)$$

to the monoid of homotopy self-equivalences of S^k determines a spherical fibration over X or a sphere bundle up to fiber homotopy equivalence . Let us denote this topological monoid by $G(k) = \mathrm{Equiv}(S^k, S^k)$. If we choose k large enough, we have an isomorphism

$$\mathcal{SF}(S^n) \cong \pi_{n-1}G(k) \text{ for } k \gg 0.$$

But we can say a bit more. An element of $G(k)$ is a map $S^k \to S^k$. Now we observe that $G(k)$ is a subset of maps of degree ± 1

$$\Omega^k_{\pm 1} S^k \subset \Omega^k S^k = \mathrm{Map}_*(S^k, S^k).$$

Therefore, if we subtract the identity, we get an isomorphism

$$\pi_{n-1}G(k) \cong \pi_{n-1+k}(S^k) \text{ for } k \gg 0.$$

Thus, the group $\mathcal{SF}(S^n)$ is equal to the $(n-1)$st stable homotopy group of the sphere

$$\mathcal{SF}(S^n) \cong \pi^s_{n-1}(S^0).$$

Hence, for $X = S^n$, the map

$$KO(S^n) \to \mathcal{SF}(S^n)$$

defined by taking fiberwise one-point compactifications *is* the J-homomorphism.

Motivated by this example, we will call the map

$$J\colon KO(X) \to \mathcal{SF}(X)$$

the *J-homomorphism* for any finite cell complex X. As a consequence of the discussion in Example 35.3 we also get the following finiteness result of Atiyah's.

Proposition 35.4. *If X is a connected finite cell complex, the group $\mathcal{SF}(X)$ is finite.*

Sketch of a proof. We can argue just as in Example 35.3 that every element in $\mathcal{SF}(X)$ is classified by a homotopy class of a map

$$X \to BG(k) \text{ for } k \gg 0$$

where $BG(k)$ denotes the classifying space of the monoid $G(k)$ (such a classifying space construction exists). Since X is a finite cell complex we can use induction on the number of cells and are reduced to show that $\pi_n BG(k)$ is finite. But the latter group is equal to $\pi_{n-1}G(k)$ and we have seen in Example 35.3 that this group is equal to $\pi^s_{n-1}(S^0)$. The stable homotopy groups of the sphere spectrum are finite by Serre. □

35.2. **The Adams conjecture.** Recall that Adams conjectured the following property of the J-homomorphism.

The Adams conjecture 35.5. *Let X be a finite cell complex, k an integer, and $y \in KO(X)$. Then there exists a non-negative integer $e = e(k, y)$ such that*

$$k^e(\psi^k - 1)y \in \operatorname{Ker} J.$$

Moreover, these elements (for all k) generate the kernel of J.

Remark 35.6. We could reformulate the assertion of the theorem as follows. For every prime p not dividing k the kernel of the map

$$KO(X)_{(p)} \to \mathcal{SF}(X)_{(p)}$$

is generated by elements of the form $(\psi^k - 1)y$.

Before we go on, let us see how the following result of Adams', used in the previous lecture for $X = S^{4n}$, follows from the first part of Conjecture 35.5. (We use the notation of the previous lecture.)

Proposition 35.7. *The group $J''(X)$ is an upper bound for the image of J in $\mathcal{SF}(X)$.*

Proof. Let $T(X)$ be the kernel of J and $Y = KO(X)$. By 35.5 there is a function $e(k, y)$ such that $Y_e \subseteq T(X)$, where Y_e is the subgroup of Y generated by all elements of the form $k^e(\psi^k - 1)y$. This shows that the intersection $\cap_e Y_e$ is contained in $T(X)$. But $J''(X)$ is by definition the quotient

$$J''(X) = Y / \cap_e Y_e.$$

So we have a surjective map $KO(X)/\cap_e Y_e \to KO(X)/T(X)$. In particular, every element in the image of J is also in the image of the induced map $J''(X) \to SF(X)$. □

35.3. **Line bundles and the mod k Dold theorem.** We will sketch a proof of Adams' conjecture in the next lecture. Today we study some special cases. We begin with an easy observation.

Remark 35.8. If the first assertion of 35.5 holds for all vector bundles of even rank, then it holds for all vector bundles. For, if ξ is a bundle of odd rank, then by assumption there is an N such that

$$k^N(\psi^k - 1)(\xi \oplus \epsilon^1) \in \operatorname{Ker} J,$$

and hence

$$k^N(\psi^k - 1)\xi = k^N(\psi^k(\xi) - \xi) + k^N(\epsilon^1 - \epsilon^1) = k^N(\psi^k(\xi \oplus \epsilon^1) - (\xi \oplus \epsilon^1)) \in \operatorname{Ker} J.$$

Proposition 35.9. *Let $y \in KO(X)$ be a linear combination of real line bundles over the finite cell complex X. Then there exists an $e \in \mathbb{N}$ (depending only on the dimension of X) such that*

$$k^e(\psi^k - 1)y = 0.$$

Proof. Since $k^e(\psi^k - 1)y$ is linear in y, it suffices to consider the case in which y is a real line bundle. In this case, since X is a finite cell complex, there exists a map $f\colon X \to \mathbb{R}\mathrm{P}^n$ for some n such that $y = f^*\gamma$, where γ is the canonical real line bundle over $\mathbb{R}\mathrm{P}^n$. Hence it suffices to prove the assertion for $y = \gamma$.

The $KO(\mathbb{R}\mathrm{P}^n)$ is a finite 2-group generated by $1-\gamma$. (If you know about spectral sequences, you can deduce this easily from the Atiyah–Hirzebruch spectral sequence and the fact that the cohomology of $\mathbb{R}\mathrm{P}^n$ is a finite 2-group.) Hence there is an $e \in \mathbb{N}$ such that

$$2^e(\psi^k - 1)y = 0.$$

If k is even, this implies $k^e(\psi^k - 1)y = 0$. If k is odd, then we have the relation $y^2 = 1$ in $KO(\mathbb{R}\mathrm{P}^n)$. This implies $\psi^k(y) = y^k = y$ and hence $(\psi^k - 1)y = 0$. To see that we have $y^2 = 1$ there are many different ways. For example, one could use the fact that real line bundles are characterized by their first Stiefel–Whitney class. Or one notices that the structure group of a real line bundle is $O(1) = \{+1, -1\}$ from which one sees $\gamma \otimes \gamma = 1$. □

The proof of Adams' conjecture 35.5 uses the following generalization of Dold's results.

Theorem 35.10 (mod k Dold theorem)**.** *Let X be a finite cell complex. Suppose there is a map of sphere bundles $\xi_1 \to \xi_2$ of the same dimension such that the map on fibers $S^n \xrightarrow{k} S^n$ is of degree k. Then there exists a non-negative integer e such that $k^e\xi_1$ and $k^e\xi_2$ are fibre homotopy equivalent and hence $k^e\xi_1 = k^e\xi_2 \in \mathcal{SF}(X)$.*

Example 35.11. Let L be a complex line bundle, or equivalently an oriented 2-dimensional real vector bundle. Then the map

$$X \to \mathbb{C}\mathrm{P}^\infty \xrightarrow{k} \mathbb{C}\mathrm{P}^\infty$$

classifies $L^{\otimes k}$. The map $\mathbb{C}\mathrm{P}^\infty \xrightarrow{k} \mathbb{C}\mathrm{P}^\infty$ is covered by a map of universal bundles which is fiberwise the degree k map. For sending L to $L^{\otimes k}$ corresponds in each fiber to the map $z \mapsto z^k$. Then the mod k Dold theorem implies that there is an e such that $k^e\psi^k(L) = k^eL^{\otimes k}$ and k^eL are fiber homotopy equivalent. Alternatively, we could say that $\psi^k(L) - L = 0 \in \mathcal{SF}(X)[k^{-1}]$.

35.4. **Sketch of Adams' proof for $X = S^{4n}$, $4n \equiv 4$ mod 8.** Let $X = S^{2n}$ such that the map

$$r\colon K(S^{2n}) \to KO(S^{2n})$$

is surjective. So given $y \in KO(S^{2n})$ there is a $z \in K(S^{2n})$ such that $y = r(z)$. Now consider the map

$$q\colon W = S^2 \times \cdots \times S^2 \to S^2 \wedge \cdots \wedge S^2 \to S^{2n}$$

Over W every vector bundle is a linear combination of complex line bundles (think of S^2 as $\mathbb{C}\mathrm{P}^1$). In particular, q^*z is such a linear combination. Therefore

$$q^*y = r(q^*z)$$

is a linear combination of oriented 2-dimensional real vector bundles. By Example 35.11 we know that there is an e such that

$$k^e(\psi^k - 1)q^*y = q^*(k^e(\psi^k - 1)y)$$

maps to zero in $\mathcal{SF}(W)$. Finally, Adams shows that the map

$$q^*\colon \mathcal{SF}(S^{2n}) \to \mathcal{SF}(W)$$

is a monomorphism. (This requires only some knowledge about the classifying space $BG(k)$ and mapping cones.)

Adams also proved the case that $y \in KO(X)$ is a linear combination of $O(1)$- and $O(2)$-bundles. The general case was later proved independently and by very different methods by Quillen–Friedlander, Quillen, Sullivan, and Becker Gottlieb. We will sketch a proof in the next lecture.

36. Sullivan's proof of the Adams conjecture

Today we will have a look at Sullivan's beautiful ideas on Galois symmetries in topology and his proof of the Adams conjecture in the complex case. We will omit a lot of details and just outline the ideas. We encourage everyone to read Sullivan's original paper and lecture notes.

36.1. **The Adams conjecture.** Let X be a connected finite cell complex. We defined $\mathcal{SF}(X)$ as the Grothendieck group of sphere bundles over X modulo fiber homotopy equivalence. Sending a vector bundle to its fiberwise one-point compactification defines the J-homomorphism

$$J\colon KO(X) \to \mathcal{SF}(X).$$

For $X = S^n$ a sphere we showed that there is a natural isomorphism

$$\mathcal{SF}(S^n) \cong \pi_{n-1}^s(S^0)$$

with the stable homotopy group of the sphere.

Our goal is to show the following result.

Theorem 36.1 (The Adams Conjecture)**.** *Let X be a finite cell complex, k an integer, and $y \in KO(X)$. Then there exists a non-negative integer $e = e(k, y)$ such that*

$$k^e(\psi^k - 1)y \in \operatorname{Ker} J.$$

Last time we defined the monoid $G(n) = \operatorname{Equiv}(S^n, S^n)$ of self-homotopy equivalences of S^n. Taking smash product with a circle defines a map $G(n) \to G(n+1)$. Moreover, since a linear self-transformation of $\mathbb{R}^k$ extends via one-point compactification to a self-homotopy equivalence of S^n, we have a canonical map $O(n) \to G(n)$. Since we study only the complex case today (though the real case follows from an analogous argument), we compose this map with $U(n) \to O(2n)$ and get a map

$$U(n) \to G(2n).$$

This map induces a map of corresponding classifying spaces

$$BU(n) \to BG(2n).$$

We denote the colimit of the $BG(n)$ over n by BG:

$$BG := \operatorname*{colim}_{n\to\infty} BG(n).$$

Overall, we obtain a commutative diagram

$$\begin{array}{ccc} BU(n) & \longrightarrow & BG(2n) \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BG. \end{array} \tag{14}$$

The space BG is the classifying space of (stable) spherical fibration (sphere bundles up to fiber homotopy equivalence). Hence the set of spherical fibrations over X is in bijection to the set of homotopy classes of maps

$$[X, BG].$$

Now the (complex) J-homomorphism $K(X) \to \mathcal{SF}(X)$ corresponds to a map

$$[X, BU] \to [X, BG]$$

which is induced by the above map of classifying spaces which we also denote by

$$J\colon BU \to BG.$$

Furthermore, the kth Adams operation corresponds to a map of classifying spaces

$$\psi^k\colon BU \to BU.$$

Now given an n-dimensional complex vector bundle E over X, its associated sphere bundle corresponds to a map

$$X \xrightarrow{E} BU(n) \xrightarrow{i} BU \xrightarrow{J} BG$$

where i is the inclusion. If we apply the kth Adams operation we get a corresponding map

$$X \xrightarrow{E} BU(n) \xrightarrow{\psi^k} BU \xrightarrow{J} BG.$$

Hence to prove the Adams conjecture we need to show that up to multiplication by some power k^e the map

$$BU(n) \xrightarrow{\psi^k - i} BU \xrightarrow{J} BG \tag{15}$$

is null-homotopic, that is homotopic to a constant map.

Let us dream about a strategy for the proof for a moment. The homotopy class of the map

$$J \circ i\colon BU(n) \to BG$$

classifies a sphere bundle up to fiber homotopy. This bundle is the sphere bundle associated to the canonical bundle γ_n over $BU(n)$. Now it turns out that this bundle is fiber homotopy equivalent to the fibration

$$BU(n-1) \to BU(n).$$

Hence we can also think of $BU(n-1)$ as the total space of the spherical fibration $J(\gamma_n)$.

Then if we had a (homotopy) pullback diagram of the form

$$\begin{array}{ccc} BU(n-1) & \xrightarrow{\psi^k} & BU(n-1) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ BU(n) & \xrightarrow{\psi^k} & BU(n) \end{array} \tag{16}$$

with self-homotopy equivalences ψ^k then we would be done. For, the diagram would show that

- the spherical fibration over $BU(n)$ classified by $J \circ \psi^k$ is the pullback of
$$i\colon BU(n-1) \to BU(n) \text{ along } \psi^k\colon BU(n) \to BU(n);$$
- and hence, since the maps ψ^k are equivalences, the sphere bundles corresponding to $J \circ i$ and $J \circ \psi^k$ are fiber homotopy equivalent.

Unfortunately, the Adams operations ψ^k are self-homotopy equivalences of BU and there is no way to produce them as operations on $BU(n)$ (at least not compatibly for all n and with all properties).

This is a bummer. But here comes Sullivan's great idea. Even though the ψ^k do not exist on the $BU(n)$, they exist on the profinite completion $\widehat{BU}(n)$. Moreover, they fit into a beautiful picture of Galois symmetries in topology. Let us have a look at how this works.

36.2. **Galois symmetries.** The crucial observation is that the homotopy groups of BG are finite (remember they are isomorphic to the stable homotopy groups of the sphere spectrum). This implies that the map $J\colon BU \to BG$ factors through the profinite completion of BU

$$\begin{array}{ccccc} & & \widehat{BU} & & \\ & \nearrow & & \searrow{\scriptstyle \hat{J}} & \\ BU & & \xrightarrow{\quad J \quad} & & BG. \end{array}$$

The space $\widehat{BU}$ is the *profinite completion of* BU, i.e., it is a space endowed with a map $BU \to \widehat{BU}$ which induces the profinite completion on homotopy groups

$$\pi_* BU \to \pi_* \widehat{BU} = (\pi_* BU)^\wedge,$$

which, in even degrees, is just the completion of the integers $\mathbb{Z} \to \hat{\mathbb{Z}}$.

We call

$$\hat{K}(X) = [X, \hat{BU}]$$

the *profinite K-theory* of X.

Remark 36.2. Such a space $\hat{BU}$ exists and Sullivan establishes a lot of interesting results about profinite homotopy. We will skip to explain how you obtain $\hat{BU}$ and omit the technical subtleties, since there is more interesting theory to explore. Another source for profinite completion in homotopy theory is the work of Artin-Mazur.

Now Sullivan shows that the map from stable fiber homotopy types to profinite stable homotopy types is injective. Hence it suffices to show that, up to multiplication by some power k^e, the induced composite map

$$BU\hat{(}n) \xrightarrow{\psi^k - i} \hat{BU} \xrightarrow{\hat{j}} BG \tag{17}$$

is null-homotopic. In fact, since we are only interested in showing that the map is null-homotopic after localizing at p, $(p,k)=1$, it suffices to consider pro-p-completions. So we consider $\hat{BU}$ as the p-completed space if necessary, even though we will omit the p in the notation. (The smarter way to handle this is to redefine the ψ^k on the profinite completion as the identity if p divides k.)

Next comes a really cool move of Sullivan's. Using algebraic geometry, in particular étale homotopy theory, he interprets the Adams operations on the profinite completion of BU as elements in the absolute Galois group of $\mathbb{Q}$ and shows that there are unstable operations ψ^k on each $BU\hat{(}n)$. This is all the more remarkable, since the ψ^k do not exist as operations $BU(n) \to BU(n)$ (if we require all the nice properties they have as self-maps of BU).

Here is the idea. We can write the complex Grassmannian $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ as a quotient

$$\mathrm{Gr}_n(\mathbb{C}^{n+k}) \cong \mathrm{GL}(n+k,\mathbb{C})/(\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(k,\mathbb{C})). \tag{18}$$

So we may consider the Grassmannian as an affine smooth complex algebraic variety (for the real Grassmannian replace $GL(-,\mathbb{C})$ with $O(-,\mathbb{C})$).

Now there is a purely algebraic way to assign to every algebraic variety V over any base field a homotopy type represented by a CW-complex. The machinery which does this is called *étale homotopy theory* and has been developed by Artin-Mazur and Friedlander. The idea is similar to computing cohomology via Čech coverings. If X is a nice topological space we can compute its cohomology by taking an open covering $U \to X$ and form the Čech nerve. If the covering is nice, i.e., if each intersection of open sets is contractible, then the cohomology of the Čech nerve is equal to the cohomology of X. Not every space admits nice coverings, but if we take the limit over all coverings, i.e., the colimit over all cohomology groups of the corresponding Čech nerves, then we still recover the cohomology of X.

Now we transport this idea to algebraic geometry. Unfortunately, there are not enough open coverings of a variety V in its intrinsic topology, the Zariski topology. But Grothendieck showed that we do not actually need a topology in the usual sense to compute cohomology, it suffices to consider maps $U \to X$ of a certain types (instead of taking open *subsets*). The correct generalization of an open subset in our case is the notion of an *étale map*. An étale map between (smooth) algebraic varieties is the analogue of a local diffeomorphism between manifolds. You should think about what that means or read about it. There is actually a criterion using Jacobian determinants which makes the analogy obvious.

So we can speak of an *étale open covering* by taking an étale surjective map $U \to V$. Now we can apply the Čech construction and form a simplicial variety $U.$ whose nth term is the $(n+1)$-fold fiber product

$$U \times_X U \times_X \cdots \times_X U$$

of U over X. Applying the connected component functor to $U.$ in each degree yields a simplicial set $\pi_0(U.)$. Taking its geometric realization gives us a CW-complex. If V is a finite-dimensional smooth variety, then this is actually a finite cell complex.

As in topology taking just one such covering is not enough to describe the homotopy type of V. But if we make the coverings finer and finer and consider the colimit over all of them (actually the cofiltering system of all such coverings), then we get the correct *profinite homotopy type*.

Remark 36.3. Using étale Čech coverings is actually sufficient for smooth quasi-projective varieties over a field. For more general schemes one has to consider hypercoverings. But that's a different story.

So let us focus on our case. What we learn from this story is that there is a purely algebraic construction of the profinite homotopy type of the Grassmannian manifold and we can write

$$\hat{\mathrm{Gr}}_n(\mathbb{C}^{n+k}) \simeq \lim_{\alpha} N_\alpha \tag{19}$$

where the N_α run through these algebraic étale coverings space (actually the associated finite cell complexes).

Now we come to the crucial point. The equations defining the Grassmannian in (18) actually have rational (in fact integer) coefficients. So we can consider the Grassmannian as a variety *defined over* $\mathbb{Q}$. Hence each automorphism σ of $\mathbb{C}$

fixing $\mathbb{Q}$ acts on the (complex) points of the Grassmannian. This is nice, though there is the problem: the action of σ is “highly discontinuous”, at least in the sense that it does not induce an interesting automorphism on cohomology.

That’s bad news. But here is the solution: Each variety N_α in (19) is defined over $\mathbb{Q}$ and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the system of the N_α’s. After taking the union over all k, *this defines an action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *on the* profinite classifying space $\hat{BU(n)}$ *(and on* $\hat{BU}$*).*

Now consider the natural surjective homomorphism

$$\chi\colon \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \to \hat{\mathbb{Z}}_p^*$$

obtained by letting $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ act on the roots of unity. (This is also called the *cyclotomic character*.)

Example 36.4. One can check that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on $\hat{BU(1)} = \hat{\mathbb{C}\mathrm{P}^\infty} = K(\hat{Z}_p, 2)$ via χ and the natural action of $\hat{Z}_p^*$ on $K(\hat{Z}_p, 2)$. (You should do this yourself after reading more about étale coverings, but you could also look it up in Sullivan’s MIT notes §5.)

From this example it follows by naturality and the splitting principle that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts through $\hat{\mathbb{Z}}_p^*$ and χ on $\hat{BU(n)}$. That means that σ acts on cohomology via

$$\sigma(c_i) = \chi(\sigma)^{-1} c_i$$

where c_i is the ith Chern class (which is a generator of the cohomology of $BU(n)$).

Proposition 36.5. *Given k in $\hat{Z}_p^*$, choose a $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ such that $\chi(\sigma) = k^{-1}$. Then*

$$\sigma\colon \hat{BU(n)} \to \hat{BU(n)}$$

is an unstable Adams operation *in the sense that the diagram*

$$\begin{array}{ccc} \hat{BU(n)} & \longrightarrow & \hat{BU} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \hat{\psi}^k} \\ \hat{BU(n)} & \longrightarrow & \hat{BU} \end{array}$$

is commutative up to homotopy. Moreover, the operations σ are compatible if n varies.

Sketch of the proof. To show that the diagram is homotopy commutative amounts to show that the elements in profinitely completed K-theory corresponding to the horizontal maps agree. For this it suffices to show by the splitting principle that the diagram

$$\begin{array}{ccc} \widehat{BU(1)} & \longrightarrow & \widehat{BU} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\hat{\psi}^k} \\ \widehat{BU(1)} & \longrightarrow & \widehat{BU} \end{array}$$

is commutative up to homotopy. But we know from the above example that σ raises elements to the kth power and this is what ψ^k does on line bundles. □

Remark 36.6. The fact that we can define Adams operations on the profinite completion $\widehat{BU(n)}$ is very remarkable, since there are no unstable Adams operations on $BU(n)$ itself. *The key is the natural Galois action on the inverse system of étale coverings.*

So Sullivan concludes that we can reformulate the Adams conjecture in the following way.

Theorem 36.7. *The stable fiber homotopy type of elements in profinite K-theory is constant on the orbits of the Galois group.*

Proof. Proposition 36.5 shows that we have a homotopy pullback diagram

$$\begin{array}{ccc} \widehat{BU(n-1)} & \xrightarrow{\psi^k} & \widehat{BU(n-1)} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \widehat{BU(n)} & \xrightarrow{\psi^k} & \widehat{BU(n)} \end{array} \tag{20}$$

where the ψ^k are given by the Galois symmetries σ and are homotopy equivalences. So for the profinite completions we can argue as we wanted that

- the completed spherical fibration over $\widehat{BU(n)}$ classified by $\hat{J} \circ \psi^k$ is the pullback of

  $$i \colon \widehat{BU(n-1)} \to \widehat{BU(n)} \text{ along } \psi^k = \sigma \colon \widehat{BU(n)} \to \widehat{BU(n)};$$

- and hence, since the maps $\psi^k = \sigma$ are equivalences, the completed sphere bundles corresponding to $\hat{J} \circ i$ and $\hat{J} \circ \psi^k$ are fiber homotopy equivalent.

This shows that the sphere bundles associated to $\hat{\gamma}_n$ and $\psi^k(\hat{\gamma}_n) = \hat{\gamma}_n^\sigma$ have the same unstable profinite homotopy types. But this implies that also the stable sphere bundles associated to $\hat{\gamma}$ and $\psi^k(\hat{\gamma}) = \hat{\gamma}^\sigma$ have the same stable profinite homotopy types. □

Remark 36.8. 1. This completes the proof of the Adams conjecture in the complex case. The argument for the real case is similar. We just have to take care of the extra information of the extension $\mathbb{C}/\mathbb{R}$.
2. The proof shows more than just the stable version in Theorem 36.7. It also proves an unstable (real and complex) profinite version of the Adams conjecture.
3. It is in fact not necessary to just complete at primes p with $(p, k) = 1$. If one redefines the Adams operations appropriately at the primes p dividing k one can take profinite completions with respect to all primes at once.

Appendix A. Slides from talks on the Adams conjecture

What follows are parts of my slides from an invited lecture series on étale homotopy theory at Heidelberg University in March 2014. I extracted and put together the parts on the Adams conjecture. I hope the slides are interesting and helpful.

Proofs of the Adams conjecture:

We will discuss two methods to prove the Adams conjecture (and there are more). Both involve etale homotopy theory in an essential way.

- Today: Quillen-Friedlander's approach.

Compare spaces over complex numbers with spaces in characteristic p and use the Frobenius map.

- In Lecture 3: Sullivan's approach.

Galois symmetries on profinite completions of spaces are induced by etale homotopy types.

Spherical fibrations:

Let X be a finite CW-complex and let E be an n-dimensional complex vector bundle over X.

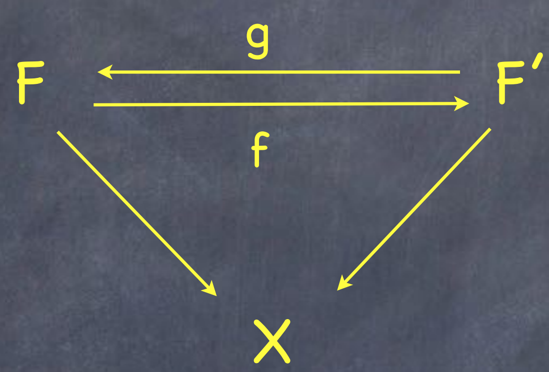
By endowing E with a Hermitean metric and looking at vectors of length 1 in $E-0$ we get a fiber bundle

$$S(E) \to X$$

with fiber a $2n-1$-sphere S^{2n-1}.

Fiber homotopy equivalence:

We say that two fiber bundles F and F' over X



are "fiber homotopy equivalent" if

there are maps f and g

and homotopy equivalences $gf \simeq id_F$ and $fg \simeq id_{F'}$ which at each time t are maps of fiber bundles.

The J-homomorphism:

Let K(X) be the Grothendieck group of finite dimensional complex vector bundles over X.

Let SF(X) be the Grothendieck group of spherical fibrations modulo fiber homotopy equivalence.

The functor S(-) induces the J-homomorphism

$$J\colon K(X) \to SF(X).$$

The Adams conjecture:

Let ψ^k be the kth Adams operation on $K(X)$. It is a functorial ring homomorphism. For a line bundle L, it is $\psi^k(L)=L^k$ in $K(X)$.

Adams' conjecture: Let E be a complex vector bundle over a finite CW-complex X and k an integer. Then there is an integer n such that $k^n(\psi^k E-E)$ maps to zero under J.

(In fact, Adams conjectures also the case of real vector bundles.)

The Quillen-Friedlander approach:

Let us assume we already knew there is a CW-complex V_{et} which represents the etale homotopy type for every reasonable scheme V.

The idea of the proof is based on three obeservations:

Quillen's observation 1:

- Homotopy types are visible in charateristic p.

Let R be a strict henselization of Z at p, $R \subset C$ an embedding and $k=\bar{F}_p$ the closed point of R, V_R a proper smooth scheme over R.

Then there are canonical equivalences of spaces

$$V\hat{}_{C,cl} \xrightarrow{\sim} V\hat{}_{C,et} \xrightarrow{\sim} V\hat{}_{R,et} \xleftarrow{\sim} V\hat{}_{k,et}$$

where ^ denotes profinite completion away from p.

Quillen's observation 2:

- Frobenius maps give Adams operations.

Let V be a scheme of characteristic p and E an algebraic vector bundle over V.

Let $F: V \to V$ be the Frobenius map and write

$$E^{(p)} = F^*E.$$

Then we have an equality in $K(V)$

$$\psi^p(E) = E^{(p)}.$$

Quillen's observation 3:

- The Frobenius identifies sphere bundles.

Let E be an algebraic vector bundle over a scheme in characteristic p.

Frobenius $E \longrightarrow E^{(p)}$ restricts to $E-0 \longrightarrow E^{(p)}-0$

and induces an equivalence

$$(E-0)\hat{}_{et} \approx (E^{(p)}-0)\hat{}_{et}.$$

The Quillen-Friedlander proof:

First of all, since $\psi^{ab} = \psi^a \psi^b$, we can assume that $k=p$ is a prime number.

It suffices to prove the conjecture for the Grassmannian $Gr=:V$ and the canonical bundle $E \rightarrow V$.

Crucial point: The Grassmannian and the canonical bundle can be defined as schemes over the integers.

Then we should be able to apply the observations in the following way:

The Quillen-Friedlander proof:

$$
\begin{array}{ccccccc}
K(V_{C,cl}) & & \longleftarrow & K(V_C) & \longleftarrow & K(V) & \longrightarrow & K(V_k) \\
\downarrow J & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) & \xrightarrow{\Theta_L} & SF(V^{\wedge}_{C,cl}) \longleftarrow & SF(V^{\wedge}_{C,et}) & \longleftarrow & SF(V^{\wedge}_{et}) & \longrightarrow & SF(V^{\wedge}_{k,et})
\end{array}
$$

Observe: An element in the kernel of Θ_L is of order p^n for some n.

It suffices to show $\Theta_L(J(\psi^p E_C - E_C)) = 0$ in $SF(V^{\wedge}_{C,cl})$.

For then we have $p^n J(\psi^p E_C - E_C) = 0$ in $SF(V_{C,cl})$.

The Quillen-Friedlander proof:

$$
\begin{array}{ccccccccc}
K(V_{C,cl}) & & \longleftarrow & & K(V_C) & \longleftarrow & K(V) & \longrightarrow & K(V_k) \\
\downarrow J & & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) & \xrightarrow{\Theta_L} & SF(V^{\wedge}_{C,cl}) & \xleftarrow{\approx} & SF(V^{\wedge}_{C,et}) & \longleftarrow & SF(V^{\wedge}_{et}) & \longrightarrow & SF(V^{\wedge}_{k,et})
\end{array}
$$

We need to show: $J(\psi^p(E_C)-E_C) = 0$ in $SF(V^{\wedge}_{C,cl})$.

By the comparison of classical and etale homotopy types, it suffices to show:

$$J(\psi^p(E_C)-E_C) = 0 \text{ in } SF(V^{\wedge}_{C,et}).$$

The Quillen-Friedlander proof:

$$
\begin{array}{ccccccc}
K(V_{C,cl}) & \longleftarrow & K(V_C) & \longleftarrow & K(V) & \longrightarrow & K(V_k) \\
\downarrow J & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) \xrightarrow{\Theta_L} SF(V^{\wedge}_{C,cl}) & \xleftarrow{\approx} & SF(V^{\wedge}_{C,et}) & \xleftarrow{\approx} & SF(V^{\wedge}_{et}) & \xrightarrow{\approx} & SF(V^{\wedge}_{k,et})
\end{array}
$$

We need to show: $J(\psi^p(E_C)-E_C) = 0$ in $SF(V^{\wedge}_{C,et})$.

Since "characteristic p sees homotopy", it suffices to show:

$$J(\psi^p(E_k)-E_k) \text{ in } SF(V^{\wedge}_{k,et}).$$

The Quillen-Friedlander proof:

$$
\begin{array}{ccccccccc}
K(V_{C,cl}) & & \longleftarrow & & K(V_C) & \longleftarrow & K(V) & \longrightarrow & K(V_k) \\
\downarrow J & & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{C,cl}) & \xrightarrow{\Theta_L} & SF(V_{\widehat{C,cl}}) & \xleftarrow{\approx} & SF(V_{\widehat{C,et}}) & \xleftarrow{\approx} & SF(V_{\widehat{et}}) & \xrightarrow{\approx} & SF(V_{\widehat{k,et}})
\end{array}
$$

We need to show: $J(\psi^p(E_k)-E_k) = 0$ in $SF(V_{\widehat{k,et}})$.

By "Frobenius = Adams operation" it suffices to show:

$J(E_k{}^{(p)}-E_k)$ in $SF(V_{\widehat{k,et}})$.

This holds by Observation 3 and we are done!

Friedlander's theorem:

There is a very difficult point we just assumed:

- If V is a scheme over R and E an algebraic vector bundle of dimension n, then

$$(E-0)\hat{}_{et} \to V\hat{}_{et}$$

is a (completed) $(2n-1)$-sphere fibration.

In his thesis, Friedlander proved that geometric and homotopy fibers behave well under etale homotopy types, thereby proved the Adams conjecture.

- Sullivan and Galois symmetries in topology:

Let us have a second look at the (complex version of the) Adams conjecture:

Let BU(n) be the Grassmannian of complex n-planes, BU be the infinite complex Grassmannian.

Let BG be the classifying space of (stable) spherical fibrations.

Sullivan and Galois symmetries in topology:

Adams: For all k, the map

$$J \circ (\psi^k - 1) : BU(n) \to BU \to BG[1/k]$$

is null-homotopic, i.e., homotopic to a constant map.

First step: As in Lecture 1, it suffices to consider the p-completed maps (for each p with $(k,p)=1$)

$$J \circ (\psi^k - 1) : BU(n)\hat{} \to BU\hat{} \to BG(S_p\hat{}).$$

Sullivan's amazing idea:

Interpret the Adams operations as "Galois symmetries" on profinitely completed homotopy types of classifying spaces.

Galois symmetries in topology:

The complex projective n-space P^n is defined over Q and we know

$$P^n(C)\hat{} \approx P^n_{et}.$$

The absolute Galois group Gal_Q of Q acts on P^n_{et} and this defines an action of Gal_Q on $P^n(C)\hat{}$.

Concretely: $\sigma \in Gal_Q$ acts on $\pi_2(P^n(C)\hat{})=Z_p$ by multiplication with $\chi(\sigma)$ where χ denotes the cyclotomoic character.

Galois symmetries in topology:

Just seen: $\sigma \in Gal_Q$ acts on $\pi_2(P^n(C)^\wedge)$ via $\chi(\sigma)$.

This is a surprising fact, since the action of Gal_Q on $P^1(C)$ is "wildly discontinuous". Only after completion we obtain a nice action.

Key fact: The etale homotopy type tells us how to read off the action on finite covers.

Galois symmetries in topology:

In the same way: There is a nice action of Gal_Q on $P^\infty(C)$^ ($\approx K(Z_p,2)$) and on $BU(n)$^:

Concretely: $\sigma \in Gal_Q$ acts on $BU(n)$^ such that

$$\sigma(c_i) = \chi(\sigma)^{-i} \cdot c_i$$

on cohomology, where c_i is the ith Chern class.

Galois symmetries in topology:

Choose $\sigma \in Gal_Q$ such that $\chi(\sigma) = k^{-1} \in Z_p^\times$. Then

$$\sigma : BU(n)\hat{} \to BU(n)\hat{} \text{ with}$$

$$\sigma(c_i) = k^i \cdot c_i.$$

Key observation: This σ is an "unstable version" of the Adams operation ψ^k. (Use splitting principle and compute the effect on line bundles.)

This is very remarkable: Without completions, ψ^k is an endomorphism of BU and not BU(n).

The conclusion of the proof:

We conclude: the diagram

$$\begin{array}{ccc} BU(n-1)\hat{} & \xrightarrow{\sigma=\psi^k} & BU(n-1)\hat{} \\ i\downarrow & & \downarrow i \\ BU(n)\hat{} & \xrightarrow[\sigma=\psi^k]{} & BU(n)\hat{} \end{array}$$

is homotopy commutative and cartesian.

Thus, twisting by ψ^k does not change the corresponding spherical fibration. This completes the sketch of Sullivan's proof of the Adams conjecture.

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