Math 231b
Lecture 17

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17. Lecture 17: Complex K-theory

From now on all vector bundles will complex vector bundles. For most of our arguments we will assume that the spaces are compact Hausdorff even though some statements may be true for more general spaces. In the following lectures we will mostly follow Atiyah's lecture notes on K-theory.

17.1. **Some basic definitions.** Let X be a space and let $\mathrm{Vect}(X)$ be the set of isomorphism classes of finite dimensional complex vector bundles. The set $\mathrm{Vect}(X)$ has the structure of an abelian semigroup under the composition of taking direct sums. We know that to any abelian semigroup A there is an associated abelian group $K(A)$ with the following universal property:

There is a semigroup homomorphism $\alpha\colon A \to K(A)$ such that if G is any group and $\gamma\colon A \to G$ any semigroup homomorphism, there is a unique homomorphism of groups $\kappa\colon K(A) \to G$ such that $\gamma = \kappa\alpha$. This determines $K(A)$ up to unique isomorphism.

There are different ways to construct $K(A)$. One way is to define $K(A)$ to be the set of pairs (a,b) in $A \times A$ modulo the following equivalence relation:

(1) $\quad (a,b) \sim (a',b')$ if there is a $c \in A$ such that $a + b' + c = a' + b + c$.

In other words,

$$K(A) = A \times A/\Delta(A),$$

where $\Delta\colon A \to A \times A$ denotes the diagonal.

Denoting the equivalence class of (a,b) by $[a,b]$ we can define the addition on $K(A)$ by

$$[a,b] + [a',b'] = [a + a',b + b']$$

The homomorphism $\alpha_A\colon A \to K(A)$ is defined by

$$a \mapsto [a,0],$$

where 0 denotes the zero element of A (which we assume to exist). The nice feature of this description of $K(A)$ is that the interchange of factors in $A \times A$ induces an inverse in $K(A)$ which makes $K(A)$ into a group.

The pair $(K(A), \alpha_A)$ is a functor of A so that if $f\colon A \to B$ is a semigroup homomorphism we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & K(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle K(f)} \\ B & \xrightarrow{\alpha_B} & K(B). \end{array}$$

Moreover, if B is a group then α_B is an isomorphism. This shows that $K(A)$ has the required universal property. Furthermore, if A is also a semiring, i.e., A is a semigroup with a multiplication that is distributive over the addition of A, then $K(A)$ is a ring with multiplication

$$[a,b] \cdot [a',b'] = aa' + bb', ab' + ba'].$$

Now if X is a space, we write $K(X)$ for the ring $K(\mathrm{Vect}(X))$, where the multiplication is given by forming tensor products of vector bundles. For $E \in \mathrm{Vect}(X)$ we will write $[E]$ for its image in $K(X)$, or also just E if there is no danger of confusion.

Before we proceed we need the following lemma.

Lemma 17.1. *Let B be a compact Hausdorff space and $\pi\colon E \to B$ be a complex vector bundle. Then there exists a complex vector bundle E' such that $E \oplus E'$ is a trivial bundle.*

Proof. From the case of real vector bundles, we know how to construct a map $g\colon E \to \mathbb{C}^\infty$ which is linear and injective on each fiber of π when B is paracompact. Since we assume here that B is compact, the construction of g shows that there is a some finite dimension N such that g factors through $E \to \mathbb{C}^N$.

This gives us a map $f\colon E \to B \times \mathbb{C}^N$. The image of f is a sub-bundle of $B \times \mathbb{C}^N$. Hence $E \to B$ is isomorphic to a sub-bundle of the trivial bundle $B \times \mathbb{C}^N$. The canonical Hermitian metric on this trivial bundle then yields a complementary sub-bundle E' such that $E \oplus E'$ is a trivial bundle. □

Our explicit description of $K(X)$ shows that every element of $K(X)$ is of the form $[E] - [F]$, where E and F are bundles over X. By the lemma, we can choose a bundle G such that $F \oplus G \cong \epsilon^n$ is a trivial bundle for some n. Then we have

$$[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\epsilon^n].$$

Thus, every element of $K(X)$ is of the form $[H] - [\epsilon^n]$.

Suppose now that E, F are such that $[E] = [F]$ in $K(X)$. Our explicit description (1) of $K(X)$ then shows that there is a bundle G such that $E \oplus G \cong F \oplus G$.

Let G' be a bundle such that $G \oplus G' \cong \epsilon^n$. Then

$$E \oplus G \oplus G' \cong F \oplus G \oplus G', \text{ so } E \oplus \epsilon^n \cong F \oplus \epsilon^n.$$

We say that two bundles are *stably equivalent*, if they become isomorphic after adding suitable trivial bundles to them. The above argument then shows:

Lemma 17.2. *We have $[E] = [F]$ in $K(X)$ if and only if E and F are stably equivalent.*

Now suppose that $f\colon X \to Y$ is a continuous map. Then

$$f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

induces a ring homomorphism

$$f^*\colon K(Y) \to K(X).$$

By one of the problems on Problem Set 2, this homomorphism depends only on the homotopy class of f.

17.2. **The periodicity theorem.** The fundamental theorem for K-theory is the periodicity theorem. It says, in particular, that for any X, there is an isomorphism between $K(X) \otimes K(S^2)$ and $K(X \times S^2)$. We will prove actually prove a more general statement which we will now explain.

Let E be a vector bundle over a space X, and let $\mathbb{P}(E)$ be the projective bundle (of complex lines) over X associated to E. If $p\colon \mathbb{P}(E) \to X$ is the projection map, $p^{-1}(x)$ is a complex projective space for all $x \in X$.

Remark 17.3. Projective spaces and bundles have the following nice property:

If V is a (complex) vector space, and W is a vector space of dimension one, then V and $V \otimes W$ are isomorphic, but not naturally isomorphic. However, taking projective spaces makes things easier.

For any non-zero element $w \in W$ the map

$$v \mapsto v \otimes w$$

defines an isomorphism between V and $V \otimes W$, and thus defines an isomorphism

$$\mathbb{P}(w)\colon \mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}(V \otimes W).$$

However, if w' is any other non-zero element of W, $w' = \lambda w$ for some non-zero complex number $\lambda \in \mathbb{C}^*$. Thus

$$\mathbb{P}(w) = \mathbb{P}(w'),$$

so the isomorphism between $\mathbb{P}(V)$ and $\mathbb{P}(V \otimes W)$ is natural.

Thus, if E is any vector bundle, and L is a line bundle, there is a natural isomorphism

$$\mathbb{P}(E) \cong \mathbb{P}(E \otimes L),$$

which concludes our remark.

If E is any vector bundle over X then each point $a \in \mathbb{P}(E)_x = \mathbb{P}(E_x)$ represents a one-dimensional subspace $H_x^* \subset E_x$. The union of all these defines a subspace

$$H^* \subset p^*E,$$

which consists of pairs of one-dimensional subspace in a fiber and a point on that line.

Lemma 17.4. *The space H^* is a sub-bundle of p^*E over $\mathbb{P}(E)$.*

Proof. The problem is local, so we may assume that E is a trivial. Then the lemma reduces to the fact that the canonical line bundle over $\mathbb{C}\mathrm{P}^n$ is a sub-bundle of the pullback of a trivial bundle. □

Remark 17.5. Note that we have met the real version of this line bundle before when we proved the splitting principle.

Definition 17.6. Now we define H to be the dual line bundle of H over $\mathbb{P}(E)$, i.e., for $\epsilon := \epsilon^1_{\mathbb{P}(E)}$ the trivial line bundle over $\mathbb{P}(E)$,

$$H := \mathrm{Hom}(H^*,\epsilon)$$

Remark 17.7. The choice of using H instead of H^* has historical reasons and is related to the use of canonical line- and quotient bundles in algebraic geometry. We will come back to this point later.

Example 17.8. Let X be compact space and let $\epsilon \oplus \epsilon$ be the sum of two trivial line bundles over X. Then

$$\mathbb{P}(\epsilon \oplus \epsilon) \cong X \times S^2,$$

since the bundle $\epsilon \oplus \epsilon$ has total space $X \times \mathbb{C}^2$, and hence

$$\mathbb{P}(\epsilon \oplus \epsilon) \cong X \times \mathbb{C}\mathrm{P}^1 \cong X \times S^2.$$

Moreover, H^* is just the pullback of the canonical complex line bundle $\gamma^1_{\mathbb{C}}$ over $\mathbb{C}\mathrm{P}^1$ to $X \times \mathbb{C}\mathrm{P}^1 \cong X \times S^2$. Hence H is the dual line bundle

$$H = \mathrm{Hom}(\gamma^1_{\mathbb{C}}, \epsilon).$$

We can now state the periodicity theorem.

Theorem 17.9. *Let X be a compact space, let L be a line bundle over X, and let $H = H(L \oplus \epsilon)$. Then, as a $K(X)$-algebra, $K(\mathbb{P}(L \oplus \epsilon))$ is generated by $[H]$, and is subject to the single relation*

$$([H] - [\epsilon])([L][H] - [\epsilon]) = 0.$$

The proof will be the topic of following lectures. Today we just point out two consequences of the theorem. The first one follows from the theorem and Example 17.8 for $X = *$ a point (and $L = \epsilon$ the trivial line bundle).

Corollary 17.10. *As a $K(*)$-module $K(S^2)$ is generated by $[H]$ and $[H]$ is subject to the single relation*

$$([H] - [\epsilon])^2 = 0.$$

The second one requires a little bit of analysis of the ring structures given by Theorem 17.9 and Corollary 17.10.

Corollary 17.11. *Let X be a compact space and*

$$\mu\colon K(X) \otimes K(S^2) \to K(X \times S^2)$$

be defined by

$$\mu(a \otimes b) = (\pi_1^* a)(\pi_2^* b),$$

where π_1 and π_2 are the projections onto the two factors. Then μ is an isomorphism of rings.

Proof. We know from Example 17.8

$$\mathbb{P}(\epsilon_X \oplus \epsilon_X) \cong X \times S^2 \text{ and } \mathbb{P}(\epsilon_* \oplus \epsilon_*) \cong S^2.$$

Under the canonical map $\pi_2\colon X \times S^2 \to * \times S^2$, the class $[H_*] \in K(S^2)$ is pulled back to the class $[H_X] \in K(X \times S^2)$. Using Theorem 17.9, we see that μ becomes the homomorphism

$$K(X) \otimes_{K(*)} K(*)[[H_*]]/(([H_*] - 1)^2) \to K(X)[[H_X]]/(([H_X] - 1)^2)$$

which by the above is just

$$K(X) \otimes_{K(*)} K(*)[[H_*]]/(([H_*] - 1)^2) \to K(X)[\pi_2^*([H_*])]/((\pi_2^*([H_*)] - 1)^2)$$

which is an isomorphism of rings. ∎